25 Solving Equations

Student Text

Contents

Section

25 Solving Equations

25.1 Algebraic Fractions

Some algebraic fractions can be simplified if the numerator and denominator are factorised.

Worked Example 1

Simplify

$$\frac{x^2 + 2x - 15}{x^2 + x - 12}$$

Solution

Both the top and bottom of the fraction can be factorised to give

$$\frac{x^2 + 2x - 15}{x^2 + x - 12} = \frac{(x+5)(x-3)}{(x+4)(x-3)}$$

The term $(x-3)$ appears as a factor on both the top and bottom of the fraction so it can be cancelled to give

$$\frac{(x+5)(x-3)}{(x+4)(x-3)} = \frac{x+5}{x+4}$$

No further simplification is possible.

Worked Example 2

Simplify

$$\frac{x}{x-1} \times \frac{x^2-1}{x^3}$$

Solution

First multiply together the fractions to give

$$\frac{x}{x-1} \times \frac{x^2-1}{x^3} = \frac{x\left(x^2-1\right)}{x^3(x-1)}$$

The bracket $\left(x^2-1\right)$ can be factorised using the difference of two squares to give

$$\frac{x\left(x^2-1\right)}{x^3(x-1)} = \frac{x(x+1)(x-1)}{x^3(x-1)}$$

Dividing numerator and denominator by common factors gives:

$$\frac{\overset{1}{\cancel{x}}(x+1)\overset{1}{\cancel{(x-1)}}}{{}_{x^2}\cancel{x^3}\,\underset{1}{\cancel{(x-1)}}} = \frac{x+1}{x^2}$$

No further simplification is possible.

Worked Example 3

Simplify

$$\frac{x+1}{x^2} \div \frac{x^2-1}{x}$$

Solution

Recall from working with fractions that

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$$

So

$$\frac{x+1}{x^2} \div \frac{x^2-1}{x} = \frac{x+1}{x^2} \times \frac{x}{x^2-1}$$

$$= \frac{x+1}{x\left(x^2-1\right)}$$

$$= \frac{x+1}{x(x+1)(x-1)}$$

$$= \frac{1}{x(x-1)}$$

Worked Example 4

Simplify

$$\frac{x}{x+2} + \frac{2x}{x+1}$$

Solution

Algebraic fractions are added in the same way as ordinary fractions, by using the lowest common denominator. In this case it will be $(x+2)(x+1)$.

$$\frac{x}{x+2} + \frac{2x}{x+1} = \frac{x(x+1)}{(x+2)(x+1)} + \frac{2x(x+2)}{(x+2)(x+1)}$$

$$= \frac{x^2 + x + 2x^2 + 4x}{(x+2)(x+1)}$$

$$= \frac{3x^2 + 5x}{(x+2)(x+1)}$$

$$= \frac{x(3x+5)}{(x+2)(x+1)}$$

No further simplification is possible.

Exercises

1. Simplify each of the following expressions.

(a) $\dfrac{x^6}{x^4}$ (b) $\dfrac{x(x+2)}{x^2(x-2)}$ (c) $\dfrac{(x+2)(x-3)}{(x-2)(x+2)}$

(d) $\dfrac{(x-4)(2x-1)}{x(x-4)(x+1)}$ (e) $\dfrac{x^3(x-6)}{x(x+6)}$ (f) $\dfrac{x(x-2)(x+3)}{x^2(x-2)(x+3)}$

(g) $\dfrac{(2x+1)(5x-3)}{x(2x+1)}$ (h) $\dfrac{(2x+1)(x-2)(x+7)}{x(x+7)(x-2)}$

(i) $\dfrac{(x+1)(x-3)(x+4)}{(x+4)(x+2)(x+1)}$

2. Simplify each of the following expressions.

(a) $\dfrac{x^2+x}{x^3+x^2}$ (b) $\dfrac{x^2+2x}{x+2}$ (c) $\dfrac{4x^2+6x}{2x+3}$

(d) $\dfrac{x^3+5x^2}{x(x+5)}$ (e) $\dfrac{x^3-x^2}{x}$ (f) $\dfrac{x^6-x^3}{x(x-1)}$

(g) $\dfrac{x^2+x}{x^2-1}$ (h) $\dfrac{x^3-4x}{x^2-3x-4}$ (i) $\dfrac{x^2-9}{x^2+4x+3}$

(j) $\dfrac{x^3-16x}{x^2-6x+8}$ (k) $\dfrac{x^2+7x+10}{x^2-x-6}$ (l) $\dfrac{x^3-6x^2+8x}{x^2-3x-4}$

(m) $\dfrac{x^3+x^2-2x}{x^4-x^3-6x^2}$ (n) $\dfrac{x^3-x}{x^2+9x+8}$ (o) $\dfrac{x^2-7x+10}{x^3+4x^2-12x}$

(p) $\dfrac{10x^2-27x+18}{2x^2-x-3}$ (q) $\dfrac{5x^2+23x-10}{x^2-2x-35}$ (r) $\dfrac{3x^2-5x-2}{5x^2-9x-2}$

(s) $\dfrac{2x^2+3x-35}{3x^2+17x+10}$ (t) $\dfrac{12x^2-5x-3}{15x^2+2x-1}$ (u) $\dfrac{9x^3-4x}{12x^2+x-6}$

3. Simplify each of the following expressions.

(a) $\dfrac{x+1}{x+4} \times \dfrac{x}{x+1}$ (b) $\dfrac{1}{(x+1)^2} \times \dfrac{x^2-1}{x}$

(c) $\dfrac{x^2+5x+6}{x} \times \dfrac{x+2}{x+3}$ (d) $\dfrac{3x}{x^2-4} \times \dfrac{x-2}{x^3}$

(e) $\dfrac{x+3}{x} \div \dfrac{x^2}{x+3}$ (f) $\dfrac{(x+3)^2}{x} \div \dfrac{x+3}{x^2}$

(g) $\dfrac{x^3}{x+6} \div \dfrac{x^2+x}{x^2+5x-6}$ (h) $\dfrac{x^3-x}{x+4} \div \dfrac{x^2+3x-4}{x^2}$

(i) $\dfrac{x^2+5x+6}{x-4} \div \dfrac{x^2-2x-8}{x+3}$

4. Express each of the following as a single fraction, simplifying if possible.

(a) $\dfrac{1}{x+2} + \dfrac{2}{x-5}$ (b) $\dfrac{4}{x+2} - \dfrac{5}{x-1}$ (c) $\dfrac{1}{x^2-1} + \dfrac{1}{x+1}$

(d) $\dfrac{3}{x+2} + \dfrac{5}{x+2}$ (e) $\dfrac{1}{x^2-9} + \dfrac{1}{x-3}$ (f) $\dfrac{4}{x-3} + \dfrac{5}{x-6}$

(g) $\dfrac{x}{x-4} - \dfrac{x}{x+2}$ (h) $\dfrac{x^2}{x+1} - \dfrac{x}{x+1}$ (i) $\dfrac{x}{x-6} + \dfrac{4}{2x-1}$

(j) $\dfrac{3x}{x+2} + \dfrac{2x}{x-6}$ (k) $\dfrac{x^2}{x^2-1} + \dfrac{x}{x+1}$ (l) $\dfrac{x+1}{x^2} + \dfrac{x+7}{x}$

(m) $\dfrac{3x}{x+6} - \dfrac{2x}{x+7}$ (n) $\dfrac{x-1}{x+2} + \dfrac{x+7}{x+3}$ (o) $\dfrac{x-3}{x+2} + \dfrac{x-7}{x+1}$

5. Simplify:

(a) $\dfrac{x^2-1}{x-1}$ (b) $\dfrac{4ab^2+2a^2b}{ab}$ *(CXC)*

25.2 Algebraic Fractions and Quadratic Equations

This section deals with finding the solutions of equations such as

$$\frac{x}{x-6} + \frac{x}{x+2} = \frac{2}{15}$$

and uses techniques from the earlier sections.

Worked Example 1

Find the solution of

$$\frac{x}{x-6} + \frac{x}{x+2} = \frac{2}{15}$$

Solution

The first stage is to combine the two fractions so that they have a common denominator.

$$\frac{x(x+2)+x(x-6)}{(x-6)(x+2)} = \frac{2}{15}$$

$$\frac{x^2+2x+x^2-6x}{(x-6)(x+2)} = \frac{2}{15}$$

$$\frac{2x^2-4x}{(x-6)(x+2)} = \frac{2}{15}$$

Then dividing both sides by 2 gives

$$\frac{x^2-2x}{(x-6)(x+2)} = \frac{1}{15}$$

Then multiplying both sides by 15 and $(x-6)(x+2)$ gives

$$15\left(x^2-2x\right) = (x-6)(x+2)$$

Therefore

$$15x^2-30x = x^2-4x-12$$

and

$$14x^2-26x+12 = 0$$

and dividing by 2 $\quad 7x^2-13x+6 = 0$

This factorises as

$$(x-1)(7x-6) = 0$$

so

$$x-1 = 0 \quad \text{or} \quad 7x-6 = 0$$

and $\quad x = 1 \quad$ or $\quad x = \frac{12}{14} = \frac{6}{7}$

Worked Example 2

Solve the equation

$$\frac{1}{x}+\frac{x}{x+2} = \frac{5}{3}$$

Solution

The first step is to combine the two fractions using the ccmmon denominator $x(x+2)$.

That is, $\quad \frac{(x+2)+x\times x}{x(x+2)} = \frac{5}{3}$

$$\frac{x^2+x+2}{x(x+2)} = \frac{5}{3}$$

Then multiplying by 3 and $x(x+2)$ gives

$$3(x^2 + x + 2) = 5x(x+2)$$

$$3x^2 + 3x + 6 = 5x^2 + 10x$$

$$0 = 2x^2 + 7x - 6$$

This quadratic can be solved by using the formula (or by completing the square) to give

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 2$, $b = 7$ and $c = -6$,

giving

$$x = \frac{-7 \pm \sqrt{49 - 4 \times 2 \times (-6)}}{2 \times 2}$$

$$= \frac{-7 \pm \sqrt{49 + 48}}{4}$$

$$= \frac{-7 \pm \sqrt{97}}{4}$$

$$x = \frac{-7 + \sqrt{97}}{4} \text{ or } \frac{-7 - \sqrt{97}}{4}$$

Exercises

Solve each of the following equations.

1. $\dfrac{x}{x+4} + \dfrac{1}{x} = \dfrac{3}{4}$

2. $\dfrac{x}{x-2} - \dfrac{x}{x+2} = \dfrac{4}{3}$

3. $\dfrac{2x}{x+1} - \dfrac{x}{x+3} = \dfrac{25}{24}$

4. $\dfrac{x}{2x+1} + \dfrac{x}{x-1} = \dfrac{12}{5}$

5. $\dfrac{x}{x+3} - \dfrac{x}{x-5} = 2$

6. $\dfrac{x}{2x+1} - \dfrac{x}{2x+7} = \dfrac{2}{15}$

7. $\dfrac{1}{x+2} + \dfrac{1}{x+3} = \dfrac{9}{20}$

8. $\dfrac{x}{x-5} - \dfrac{x}{x+5} = \dfrac{4}{3}$

9. $\dfrac{8x}{x+3} - \dfrac{x}{x-2} = 1$

10. $\dfrac{2}{x} - \dfrac{1}{x+6} = \dfrac{7}{8}$

11. $\dfrac{4}{x} + \dfrac{3}{10-x} = 2$

12. $\dfrac{1}{2x+3} + \dfrac{1}{6-x} = \dfrac{2}{5}$

13. $\dfrac{x}{x+2} - \dfrac{x-1}{x} = 4$

14. $\dfrac{1}{10-x} + \dfrac{x}{x+4} = \dfrac{4}{7}$

15. $\dfrac{3}{x+1} + \dfrac{7}{x-6} = 5$

25.3 Algebraic Solution of Simultaneous Equations – One Linear and One Quadratic

In Section 23.1 you have solved simultaneous equations where both of the equations are *linear*. In this section we extend this to solving simultaneous equations where one equation is linear and the other is *quadratic*. This will normally give you a quadratic equation to solve.

Worked Example 1

Solve the simultaneous equations

$$y = x^2 - 1 \qquad (1)$$

$$y = 5 - x \qquad (2)$$

Solution

Substituting y from equation (1) into equation (2) gives

$$x^2 - 1 = 5 - x$$

This equation simplifies to

$$0 = x^2 - 1 - 5 + x$$

so $\quad 0 = x^2 + x - 6$

We now solve this quadratic equation by factorisation.

$$0 = (x+3)(x-2)$$

so $\quad x + 3 = 0 \text{ or } x - 2 = 0$

therefore $\quad x = -3 \text{ or } x = 2$

These values of x are now substituted into one of the original equations to find the corresponding values of y. It is usually easiest to use the linear equation for this.

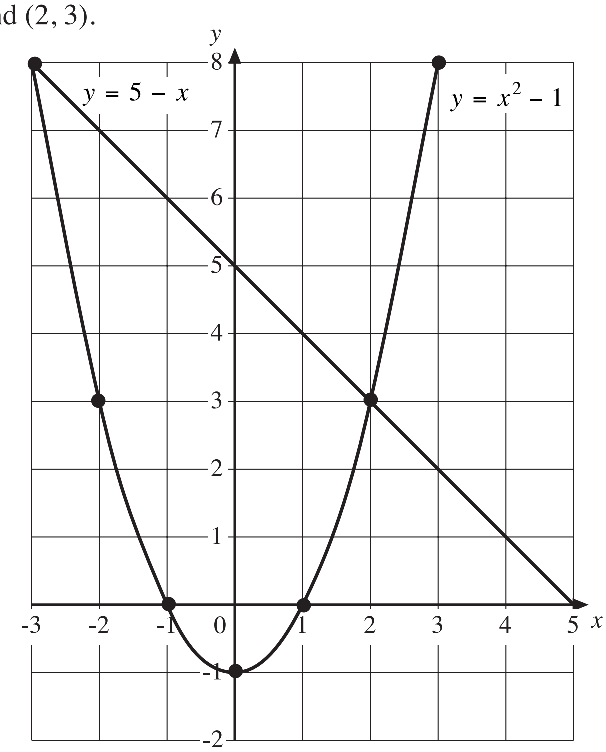
Substituting the first solution $x = -3$ into equation (2) gives $y = 5 - (-3) = 8$.

Substituting the second solution $x = 2$ into equation (2) gives $y = 5 - 2 = 3$.

The solutions are $x = -3$, $y = 8$ and $x = 2$, $y = 3$.

Note

In Worked Example 1, the first equation, $y = 5 - x$, can be represented by a straight line graph. The second equation, $y = x^2 - 1$, can be represented by a quadratic curve. When we solve this pair of simultaneous equations, we are finding the coordinates of the two points where the line and the curve intersect. This is shown on the graph, the intersection points being (–3, 8) and (2, 3).



Worked Example 2

Solve the simultaneous equations

$$y = 3x^2 - 4 \qquad (1)$$

$$y = 2x + 3 \qquad (2)$$

Solution

Subtract equation (2) from equation (1).

$$y = 3x^2 - 4 \qquad (1)$$

$$y = 2x + 3 \qquad (2)$$

$$0 = \left(3x^2 - 4\right) - \left(2x + 3\right) \qquad (2) - (1)$$

This equation simplifies to

$$0 = 3x^2 - 2x - 7$$

which does not factorise, so we need to use the quadratic equation formula to find the two possible values of x.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 3 \times (-7)}}{2 \times 3} = \frac{2 \pm \sqrt{4 + 84}}{6} = \frac{2 \pm \sqrt{88}}{6}$$

so $x = 1.896805253$

$= 1.90$ (to 2 d.p.)

or $x = -1.230138587$

$= -1.23$ (to 2 d.p.)

Substituting the un-rounded values for x into the linear equation (2) gives

$$y = 2x + 3 = 2 \times 1.896805253 + 3 = 6.793610507 = 6.79 \text{ (to 2 d.p.)}$$

and $y = 2x + 3 = 2 \times -1.230138587 + 3 = 0.539722826 = 0.54$ (to 2 d.p.).

The solutions are $x = 1.90$, $y = 6.79$ and $x = -1.23$, $y = 0.54$ (to 2 d.p.).

Worked Example 3

Solve the simultaneous equations

$$y = x^2 + 2x - 3 \qquad (1)$$

$$y = 2x \qquad (2)$$

Solution

Subtract equation (2) from equation (1).

$$y = x^2 + 2x - 3 \qquad (1)$$

$$y = 2x \qquad (2)$$

$$0 = \left(x^2 + 2x - 3\right) - \left(2x\right) \qquad (2) - (1)$$

This equation simplifies to

$$0 = x^2 - 3$$

so $x^2 = 3$

hence $x = \pm\sqrt{3}$

These values for x are now substituted into the linear equation.

Substituting the first solution, $x = \sqrt{3}$, into equation (2) gives $y = 2\sqrt{3}$.

Substituting the second solution, $x = -\sqrt{3}$, into equation (2) gives $y = 2\left(-\sqrt{3}\right) = -2\sqrt{3}$.

The solutions are $x = \sqrt{3}$, $y = 2\sqrt{3}$ and $x = -\sqrt{3}$, $y = -2\sqrt{3}$.

Worked Example 4

(a) Solve algebraically the simultaneous equations

$$x^2 + y^2 = 10 \qquad (1)$$

$$y = x + 2 \qquad (2)$$

(b) Write down the coordinates of the points of intersection of the circle $x^2 + y^2 = 10$ and the straight line $y = x + 2$.

Solution

(a) We use the linear equation (2) to substitute for y^2 into equation (1), the equation for the circle.

The square of equation (2) is

$$y^2 = (x + 2)^2$$

so $$y^2 = x^2 + 4x + 4$$

Substituting into equation (1) now gives

$$x^2 + (x^2 + 4x + 4) = 10$$

which simplifies to $$2x^2 + 4x - 6 = 0$$

which, dividing by 2, simplifies again to

$$x^2 + 2x - 3 = 0$$

We can now solve this equation for x by factorising

$$(x + 3)(x - 1) = 0$$

so $$x = -3 \text{ or } x = 1$$

When $x = -3$, equation (2) gives

$$y = -3 + 2 = -1$$

When $x = 1$, equation (2) gives

$$y = 1 + 2 = 3$$

Therefore the solutions of the two equations are

$$x = -3,\ y = -1 \text{ and } x = 1,\ y = 3$$

(b) The points of intersection of the circle $x^2 + y^2 = 10$ and the straight line $y = x + 2$ are $(-3, -1)$ and $(1, 3)$.

Exercises

1. Solve the simultaneous equations

(a) $y = x^2 - 4$
$y = 3x$

(b) $y = x^2 + 5x$
$y = 2x + 10$

(c) $y = 2x^2 + x - 3$
$y = 3x + 1$

2. Solve the following simultaneous equations, giving the values of x and y correct to 2 decimal places.

(a) $y = x^2$
$y = 3x - 1$

(b) $x = t^2 + 8$
$x = 5 - 6t$

(c) $y = 2x^2 + 12x - 4$
$y = 2x - 1$

3. Solve the simultaneous equations

(a) $x + 2y = 2$
$x^2 + 8y = 8$

(b) $4x + y = 19$
$5x^2 + 8y = 101$

(c) $4x + y = 18$
$5x^2 + 2y = 57$

4. (a) Copy and complete the table below.

x	–3	–2	–1	0	1	2	3	4
$y = 9 - x^2$								

(b) Draw the graph of $x^2 + y = 9$ for the domain $-3 \le x \le 4$, using a scale of 2 cm to 1 unit on the horizontal axis and 1 cm to 1 unit on the vertical axis.

(c) Draw the graph $y = 9 - x^2$.

(d) On the same graph, draw the straight line $y = 6 - 2x$.

(e) Write down the coordinates of the points where the line $y = 6 - 2x$ intersects the curve $y = 9 - x^2$.

(f) Use an algebraic method to solve the simultaneous equations

$$y = 9 - x^2$$
$$y = 6 - 2x$$

(g) Use the solutions found in part (f) to confirm your answer to part (e).

5. Solve the following pairs of simultaneous equations. Where appropriate, give your solutions correct to 2 decimal places.

(a) $x^2 + y^2 = 25$
$y = 3x - 5$

(b) $x^2 + y^2 = 45$
$y = 2x$

(c) $x^2 + y^2 = 8$
$y = x + 4$

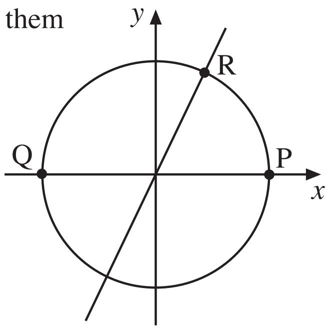
(d) $x^2 + y^2 = 68$
$y = 3x + 2$

(e) $x^2 + y^2 = 25$
$7y = -x - 25$

(f) $x^2 + y^2 = 20$
$y = 3 - 2x$

6. Find the coordinates of the points where the line $y = 4 - x$ intersects the circle $x^2 + y^2 = 22$.

7. (a) Draw the centrally-placed x- and y-axes and scale them using the ranges $-12 \le x \le 12$, $-12 \le y \le 12$.



(b) Accurately construct the locus $x^2 + y^2 = 100$.

(c) On the same graph, draw the line $y = 3x$.

(d) Write down the coordinates of the points P and Q (as labelled in the diagram) where the x-axis meets the circle. PQ is a diameter of the circle.

(e) By solving the simultaneous equations

$$x^2 + y^2 = 100$$

$$y = 3x$$

find the x- and y-coordinates of the point, R, where the line meets the circle in the first quadrant.

(f) Calculate the area of ΔPQR.

(g) Show that the gradient of the line segment QR is $\dfrac{3}{\sqrt{10} + 1}$.

(h) Find the gradient of the line segment PR.

(i) Show that PR is perpendicular to QR.

N.B This illustrates the fact that the angle in a semicircle is a right angle.

Information

Both linear and quadratic equations have been used for over four thousand years. The early Chinese and Babylonians made use of equations to solve daily problems such as the sharing of an inheritance.