Completing the square

Completing the square is a technique used to analyze quadratic functions without drawing them. The method of completing the square involves literally making a perfect square out of a given quadratic function $ax^2 + bx + c$ writing it in the form $a(x+h)^2 + q$

The process/method of completing the square is as follows using the complete method without any shortcuts.

Method 1 Full

1. separate the x terms in a bracket $(ax^2 + bx) + c$
2. divide the bracket by the coefficient of x^2 to get $a\left(x^2 + \frac{b}{a}x\right) + c$
3. halve the coefficient of the x, square it and add it to the bracket and subtract it from the constant at the same time to get $a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c - a\left(\frac{b}{2a}\right)^2$
4. Factorize the entire bracket to get $a\left(x + \frac{b}{2a}\right)^2 + c - a\left(\frac{b}{2a}\right)^2$

The whole thing looks like this

$$ax^2 + bx + c$$

$$a(x+h)^2 + q$$

$$(ax^2 + bx) + c$$

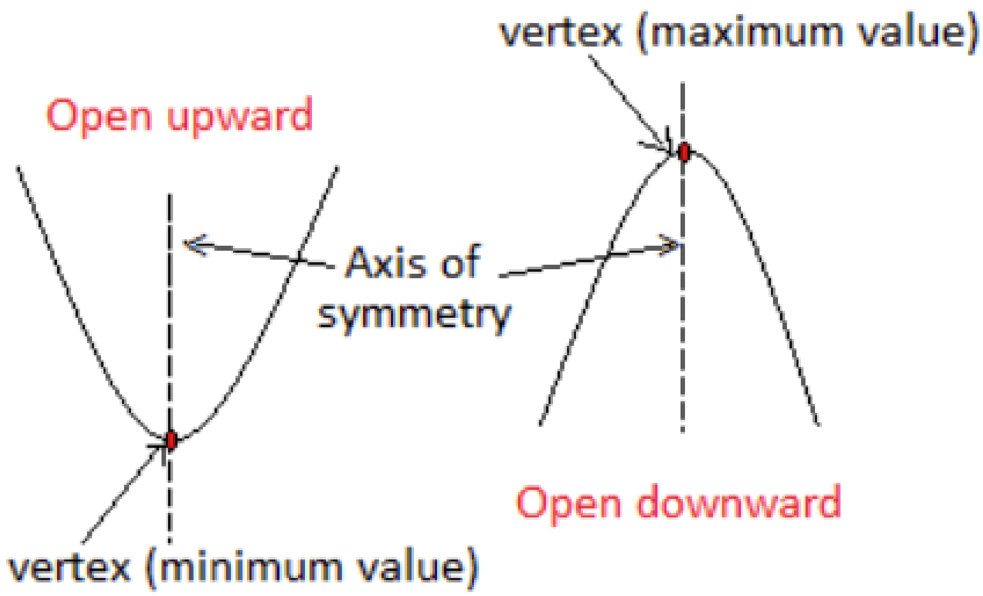
$$a\left(x^2 + \frac{b}{a}x\right) + c$$

$$a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c - a\left(\frac{b}{2a}\right)^2$$

$$a\left(x + \frac{b}{2a}\right)^2 + c - a\left(\frac{b}{2a}\right)^2$$

Comparing $a(x+h)^2 + q$ and the actual result $a\left(x + \frac{b}{2a}\right)^2 + c - a\left(\frac{b}{2a}\right)^2$ you will notice that $h = \frac{b}{2a}$ and also that $q = c - a\left(\frac{b}{2a}\right)^2$. We may complete the square directly by just calculating these values directly using the formula given ad use those values to write the given function in the form $a(x+h)^2+q$

What do the values ***a***, ***h*** and ***q*** represent?



The sign of the a value tells if the function has a minimum or maximum value

- If a is a negative number the function will have a maximum value
- If a is a positive number the function will have a minimum value

The q value is either the maximum or minimum value depending on what sign a has

The opposite of the h value is our axis of symmetry so that $-\frac{b}{2a}$

Examples

Express $f(x) = x^2 + 6x + 5$ in the form $a(x+h)^2 + q$

$f(x) = x^2 + 6x + 5$

$(x^2 + 6x) + 5$

$(x^2 + 6x + 3^2) + 5 - 3^2$

$(x^2 + 6x + 9) + 5 - 9$

$(x+3)^2 - 4$

The value $h = \dfrac{b}{2a}$ = $\dfrac{6}{2\times 1} = 3$ and =

$q = c - a\left(\dfrac{b}{2a}\right)^2$

$= 5 - 1\times 3^2 = 5 - 9 = -4$ rewriting as we

$a(x+h)^2 + q$ have $(x+3)^2 - 4$

Curve sketching

We could also sketch the curve of the following function by

1. Solving the equation
2. deciding the nature and coordinates of its turning point

$$(x+3)^2 - 4 = 0 \Rightarrow (x+3)^2 = 4$$

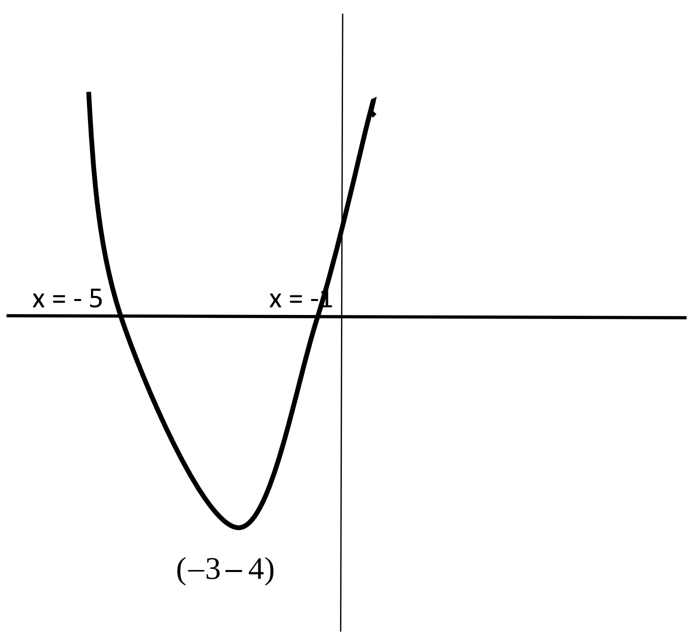
Solving gives $\sqrt{(x+3)^2} = \sqrt{4}$

$$x + 3 = +2 \Rightarrow x = 2 - 3 \Rightarrow x = -1$$

$$x + 3 = -2 \Rightarrow x = -2 - 3 \Rightarrow x = -5$$

Since the a has a positive sign (1) it means that our function has a minimum turning point of $(-3 -4)$

That's all we need to sketch, now the sketch



Express $f(x) = 4x^2 - 3x - 5$ in the form $a(x+h)^2 + q$

$$f(x) = 4x^2 - 3x - 5$$

$$\left(4x^2 - 3x\right) - 5$$

$$4\left(x^2 - \frac{3}{4}x\right) - 5$$

$$4\left(x^2 - \frac{3}{4}x + \left(-\frac{3}{8}\right)^2\right) - 5 - \left(-\frac{3}{8}\right)^2$$

$$4\left(x - \frac{3}{8}\right)^2 - 5 - \left(-\frac{3}{8}\right)^2 \times 4$$

$$4\left(x - \frac{3}{8}\right)^2 - \frac{89}{16}$$

Alternate Method via direct calculation of the values

The value $h = \frac{b}{2a}$ = $\frac{-3}{2 \times 4} = \frac{-3}{8}$ and =

$$q = c - a\left(\frac{b}{2a}\right)^2$$

$= -5 - 4 \times \left(\frac{3}{8}\right)^2 = -5 - 4 \times \left(\frac{9}{64}\right)$ rewriting as we

$$= \frac{-89}{16}$$

$a(x+h)^2 + q$ have $4\left(x - \frac{3}{8}\right)^2 - \frac{89}{16}$

Express $f(x) = 10 - 3x - 5x^2$ in the form $a(x+h)^2 + q$

$$f(x) = 10 - 3x - 5x^2 \Rightarrow -5x^2 - 3x + 10$$

$$\left(-5x^2 - 3x\right) + 10 \Rightarrow -5\left(x^2 + \frac{3}{5}x\right) + 10$$

$$-5\left(x^2 + \frac{3}{5}x + \left(\frac{3}{10}\right)^2\right) + 10 - (-5)\left(\frac{3}{10}\right)^2$$

$$-5\left(x^2 + \frac{3}{10}\right)^2 + 10.45$$

The value $h = \frac{b}{2a}$ = $\frac{-3}{2 \times -5} = \frac{3}{10}$ and =

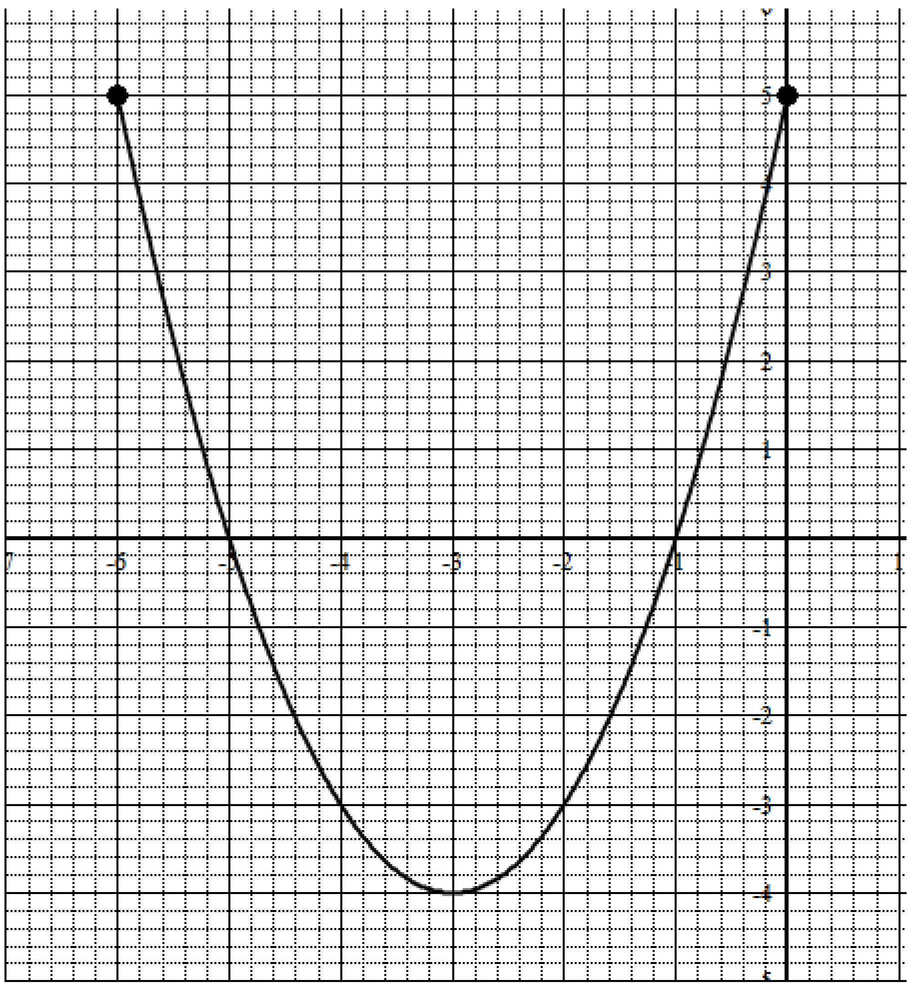
$$q = c - a\left(\frac{b}{2a}\right)^2$$

$= 10 - (-5) \times \left(\frac{-3}{10}\right)^2 = 10 - (-0.45) = 10.45$ rewriting

as we $a(x+h)^2 + q$ have $-5\left(x^2 + \frac{3}{10}\right)^2 + 10.45$

Looking at our results along with the graphs of the functions

Consider the graph of $f(x) = x^2 + 6x + 5$ and it result in the form $a(x+h)^2 + q$ as $(x+3)^2 - 4$ where $h = 3$ and $q = -4$

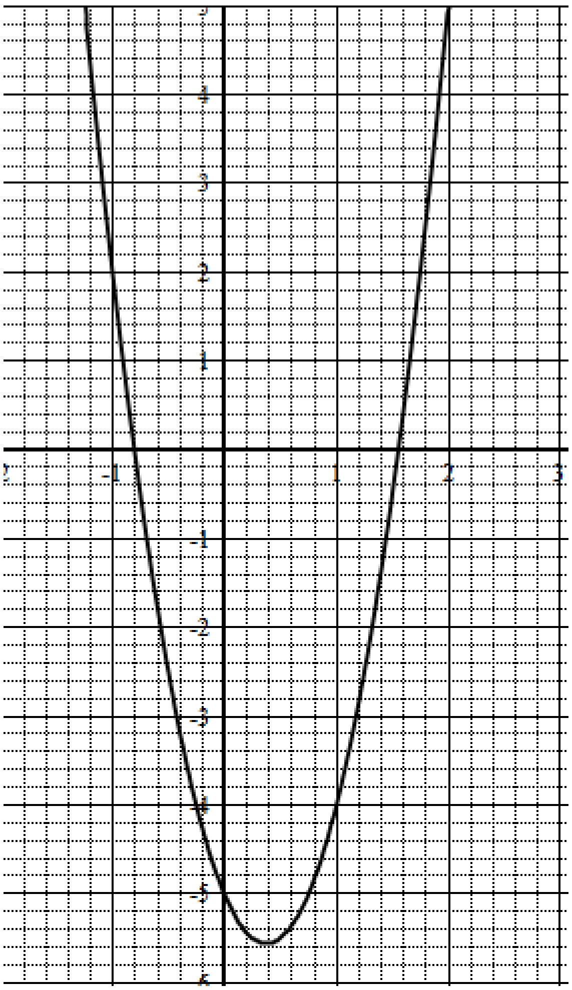


I. Notice that the minimum value reached on the graph is – 4, the same as $q = -4$, the q value.

II. Notice the axis of symmetry of the graph is $x = -3$, the opposite sign of the value $h = 3$

III. Notice also that the coordinates of the turning point are $(-3, -4)$, a combination the axis and minimum value

Consider the graph of $f(x) = 4x^2 - 3x - 5$ and



its $a(x+h)^2 + q$ which is $4\left(x - \frac{3}{8}\right)^2 - \frac{89}{16}$

$$4\left(x - \frac{3}{8}\right)^2 - \frac{89}{16} \Rightarrow 4\left(x - \frac{3}{8}\right)^2 - 5.6$$

We can clearly see that the minimum value is – 5.6, same as our ***q*** value

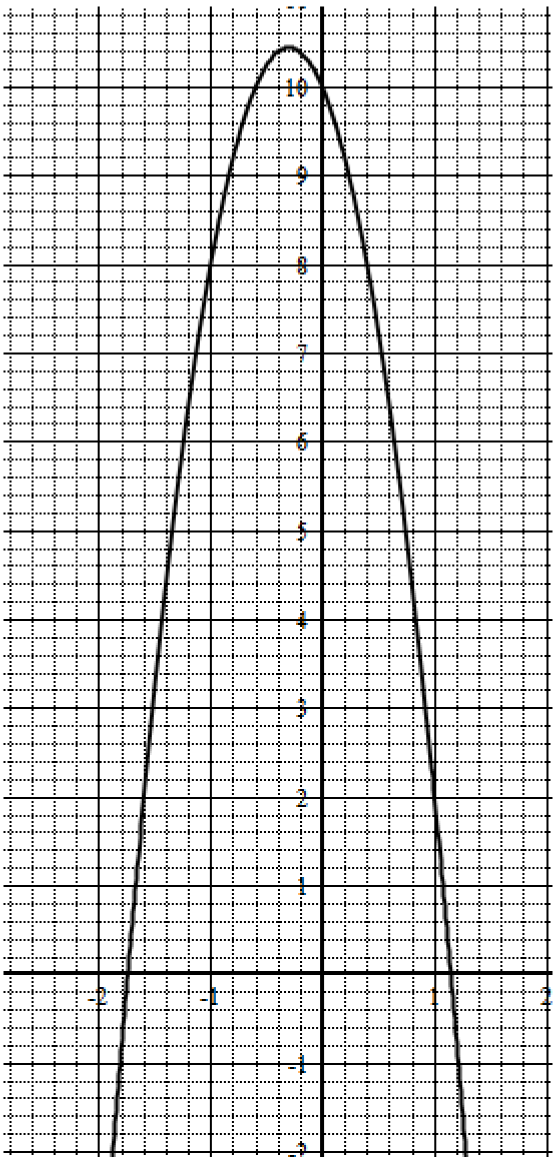
The axis of symmetry is approximately 0.4, the opposite of our ***h*** value

The coordinates of our turning point are

$$\left(\frac{3}{8}, -\frac{89}{16}\right) \Rightarrow (0.4, -5.6)$$

Finally we consider the graph of $f\left(x\right)=10-3x-5x^2$ and examine its form $a\left(x+h\right)^2+q$ which is

$$-5\left(x^2+\frac{3}{10}\right)^2+10.45$$



We can begin comparing the values immediately and we notice that **q = 10.45**, that value is the maximum value that the function reaches

The axis of symmetry in this case is the value **x = - 0.3**

This again is the opposite sign as the h value.

and we can combine these two values to get the coordinates of our maximum turning point which is

$$\left(-0.3,10.45\right)$$

Practice Questions

1. Express the quadratic function
   a. $2x^2 - 3x - 20$ in the form $a(x+h)^2 + q$ where a, h and k are constants
   b. Hence state
      i. The maximum value of $2x^2 - 3x - 20$
      ii. The equation of the axis of symmetry
      iii. Write down the coordinates of the turning point
      iv. The roots of $2x^2 - 3x - 20 = 0$ giving your answers correct to 2 decimal places

2. Write the function $f(x) = -3x^2 + 2x - 6$ in the form $a(x-b)^2 + c$ where $a, b, c \in R$ [*Real numbers*]
   a. State whether the function will have a maximum or a minimum value and write down this value
   b. What is the value of x at which the maximum or minimum occurs
   c. Hence or otherwise determine the solutions of the equation $-3x^2 + 2x - 6 = 0$

3. Given $f(x) = 2x^2 + 3x - 5$
   a. Write $f(x)$ in the form $f(x) = a(x+b)^2 + c$ where $a, b, c \in R$
   b. State the equation of the axis of symmetry
   c. State the coordinates of the minimum point in the form (x, y)
   d. Solve the equation $2x^2 + 3x - 5 = 0$
   e. Sketch the graph of $f(x)$ clearly showing the minimum point, the axis of symmetry and the y intercept