Coordinate Geometry

Coordinate geometry is the study of the relationships between points on the Cartesian plane

<u>What we will explore in this tutorial</u>

(a) Explore gradient
   I. Identify the gradient of a straight line
   II. Calculate the gradient of a straight line
   III. Determine the gradient of straight lines that are parallel to or perpendicular to a given line
(b) Calculate the midpoint of a line/line segment
(c) Calculate the length of a given line
(d) Determine the equation of
   I. A straight line
   II. The equation of a line parallel to a given line
   III. The equation of a line perpendicular to a given line
(e) Interpret the x and y intercepts of a given straight line

Gradient

Gradient may be described as a "rate of change" that is we examine how one thing is changing as the other thing is changing; for example we may heat water and compare the temperature as time passes or we may compare the distance travelled by a car compared with time.

Identifying the gradient from the equation of a straight line

The general form of a straight line is $ax + by + c = 0$, however a more popular version of this is what we call the slope intercept form of a straight line $y = mx + c$. Much of our work here will be concentrated on this form of the line

The letter m, the coefficient of x, represents our gradient. For straight lines the gradient is always constant for the whole line. You should be able to look at a straight line and easily identify its gradient; examine the equations below;

Examples of the slope intercept form are

$y = 5x - 4$; Here the gradient is 5 [**our c value or the y intercept is – 4**]

$y = 8 - 3x$ Here the gradient is – 3 [**our c value or the y intercept is 8**]

$y = \frac{2}{3}x - 4$; Here the gradient is $\frac{2}{3}$ **[our c value or the y intercept is – 4]**

$y = 8 - \frac{1}{2}x$; Here the gradient is $-\frac{1}{2}$ **[our c value or the y intercept is 8]**

In some cases however a question may give you the general form of a straight line and ask you to determine its gradient for example

1. $2y = 7x - 5$
2. $5x - 3y = 4$
3. $10 - 2x + 3y = 0$

In each case to get our answer we need to rewrite it in the form $y = mx + c$ so that we can easily see the value of our gradient.

Example 1.

State the gradient of the line $2y = 7x - 5$

$$2y = 7x - 5$$

Solution $y = \frac{7x}{2} - \frac{5}{2} \Rightarrow y = mx + c$

$$m = \frac{7}{2}$$

Example 2

Write down the gradient of $5x - 3y = 4$

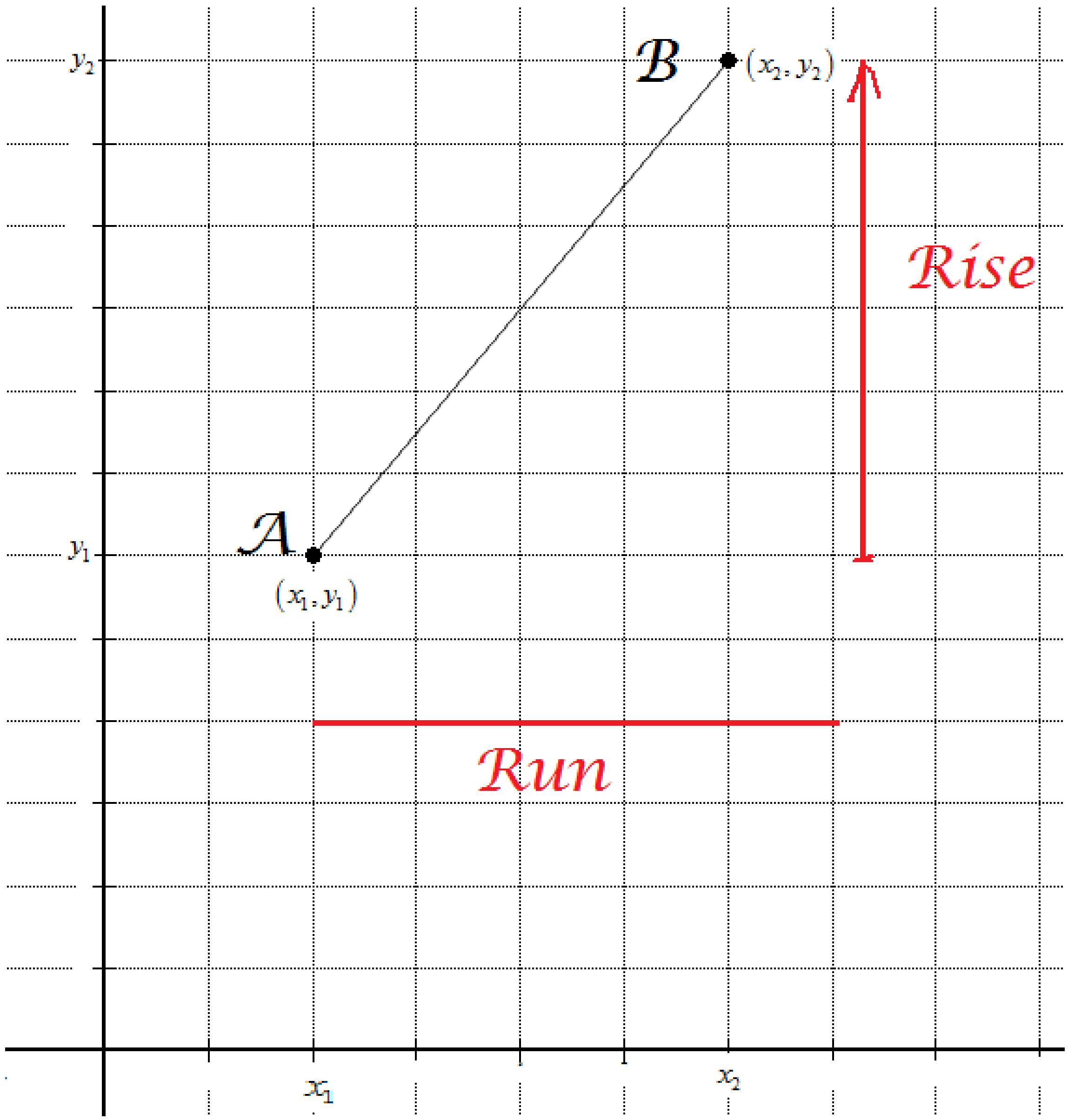
$$5x - 3y = 4$$
$$-3y = 4 - 5x$$
Solution $y = \frac{4}{-3} - \frac{5x}{-3}$
$$y = \frac{4}{-3} + \frac{5x}{3} \Rightarrow m = \frac{5}{3}$$

Example 3

Determine the gradient of $10 - 2x + 3y = 0$

$$10 - 2x + 3y = 0$$
$$3y = 2x - 10$$
Solution $y = \frac{2x - 10}{3} \Rightarrow y = \frac{2x}{3} - \frac{10}{3},$
$$\Rightarrow m = \frac{2}{3}$$

Calculate the gradient of a straight line given two pairs of coordinates $(x_1, y_1)(x_2, y_2)$



To determine the gradient of line AB we need to examine the ratio of the change in the y - distance compared with the change in the x – distance. We call the change in y the rise and the change in x the run. This can be written down as

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

> **Know this formula/concept.**
>
> This formula is used to calculate the gradient of a straight line given two points $(x_1, y_1)(x_2, y_2)$

Examples

Find the gradient of the line passing through the points given

1. $A(5,6), B(0,4)$
2. $W(6,-2), X(-2,3)$
3. $M(3,13), N(4,18)$

Solution to 1

Using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ we have $m = \frac{4-6}{0-5} = \frac{-2}{-5} = \frac{2}{5}$ note that

$(x_1 = 5, y_1 = 6)(x_2 = 0, y_2 = 4)$ or if you choose to use them alternately then

$(x_1 = 0, y_1 = 4)(x_2 = 5, y_2 = 6)$

Solution to 2

The gradient of **WX** is given as

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{3-(-2)}{-2-6} = \frac{5}{-8}$$

Solution to 3

The gradient of **MN** is given as

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

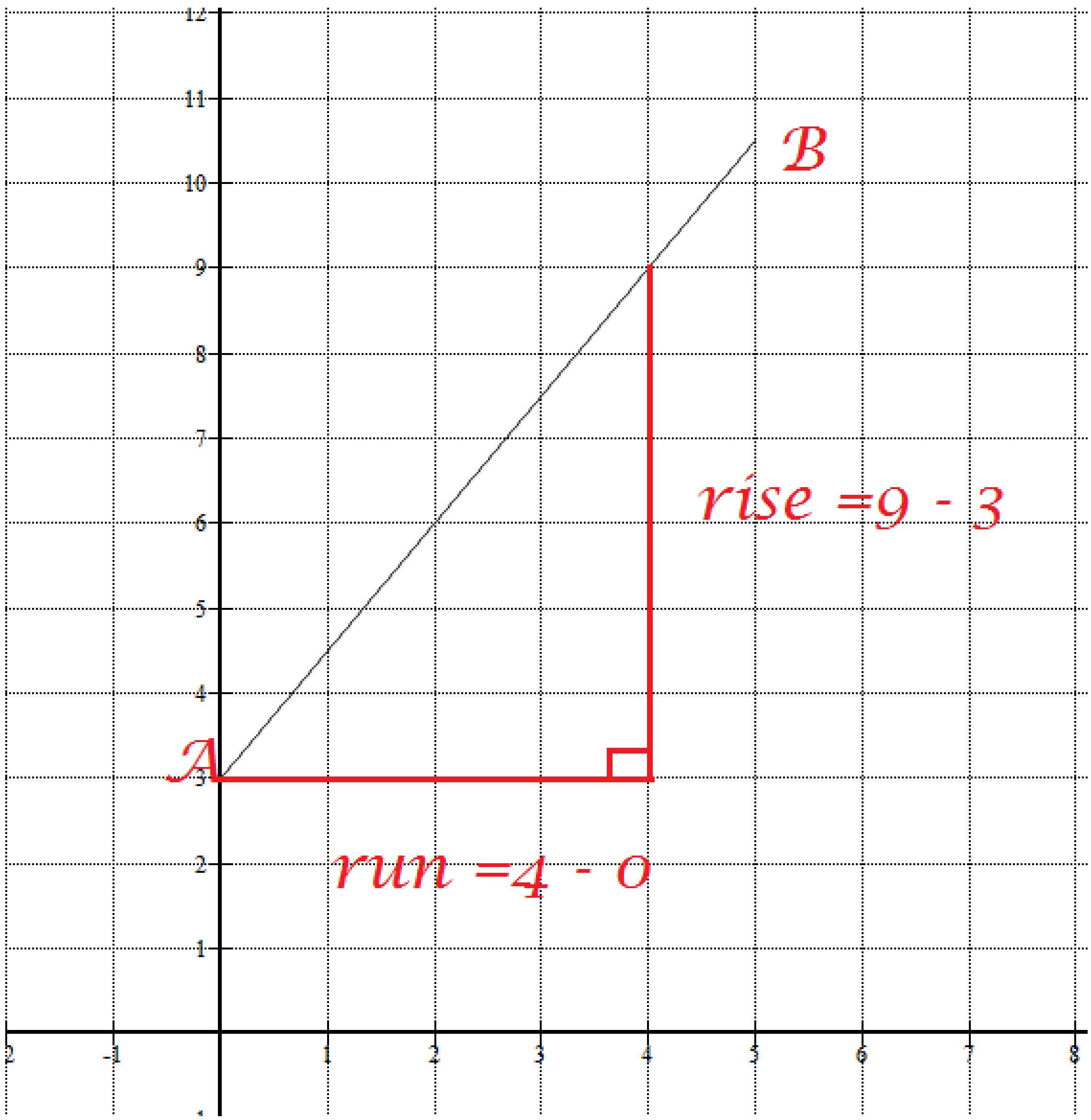
$$m = \frac{18-13}{4-3} = \frac{5}{1} = 5$$

Note.

It may help to label each coordinate individually as x_1, y_1, x_2, y_2

And use it as a guide to substitute the numbers correctly until you build up a rhythm

Finding the gradient of a straight line given its graph



The process for determining the gradient of the graph is to

1. Draw a suitable right angled triangle on the line
2. Determine the rise and the run
3. Divide the rise by the run

So we have the gradient of the line as $m = \frac{Rise}{Run} = \frac{9-3}{4-0} = \frac{6}{4}$

Parallel and perpendicular lines

Two lines are parallel if they have the same gradient
Two lines are perpendicular if the product of their gradients is – 1 [negative 1]

Examples

1. A line has the equation $y = 5x - 3$, write down the gradient of the line that is
    a. Parallel to $y = 5x - 3$
    b. Perpendicular to $y = 5x - 3$

Solution: Note that the gradient of $y = 5x - 3$ is 5 and therefore

(a) The equation of any line parallel to $y = 5x - 3$ will have a gradient of 5

(b) If two lines are perpendicular the product of their gradients is negative ONE, therefore, we can use a simple equation to find it such as $5m = -1$, $m = \frac{-1}{5}$, Note that $5 \times \frac{-1}{5} = -1$, so the gradient we need is $m = \frac{-1}{5}$. Note that $5 = \frac{5}{1}$ so we invert $\frac{5}{1}$ and change its sign to get $m = \frac{-1}{5}$

We could have also found this number $m = \frac{-1}{5}$ by inverting our gradient and changing its sign.

2. A straight line PQ has the equation $y = 4 - \frac{2}{3}x$, determine
    a. The gradient of any line that is parallel to PQ
    b. The gradient of the any line perpendicular to PQ

Solution

Our gradient here is $\frac{-2}{3}$ so

(a) Any line parallel to PQ will have a gradient of $\frac{-2}{3}$

(b) And using the explanations given above Any line perpendicular to $\frac{-2}{3}$ will have a gradient of $m=\frac{3}{2}$, we invert the $\frac{-2}{3}$ and change its sign

The midpoint and length of a line segment

There are two formulae that we need here, that is given any two points $(x_1, y_1)(x_2, y_2)$

$$midpt=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

To find midpoint

$$L=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

To find the length

Example

A straight line passes through the points $J(6,-2), K(-5,3)$ determine

(a) The midpoint of JK
(b) The length of line segment JK

The midpoint is given as $\left(\frac{6+(-5)}{2},\frac{(-2)+3}{2}\right)\Rightarrow\left(\frac{1}{2},\frac{1}{2}\right)$

$$L=\sqrt{((-5)-6)^2+(3-(-2))^2}$$

The length is given as $L=\sqrt{(-11)^2+5^2}$

$$L=\sqrt{146}=12.1units$$

Finding the equation of a straight line

Case 1; given two points $(x_1, y_1)(x_2, y_2)$

A straight line LM passes through the points $L(4,6), M(6,10)$, find the equation of LM

First we need to find the gradient which here is $m = \frac{10-6}{6-4} = \frac{4}{2} = 2$

Now using the general form of the line $y - y_1 = m(x - x_1)$ and the point $L(4,6)$ we have the equation

$$y - y_1 = m(x - x_1)$$
$$y - 6 = 2(x - 4)$$
$$y - 6 = 2x - 8$$
$$y = 2x - 8 + 6$$
$$y = 2x - 2$$

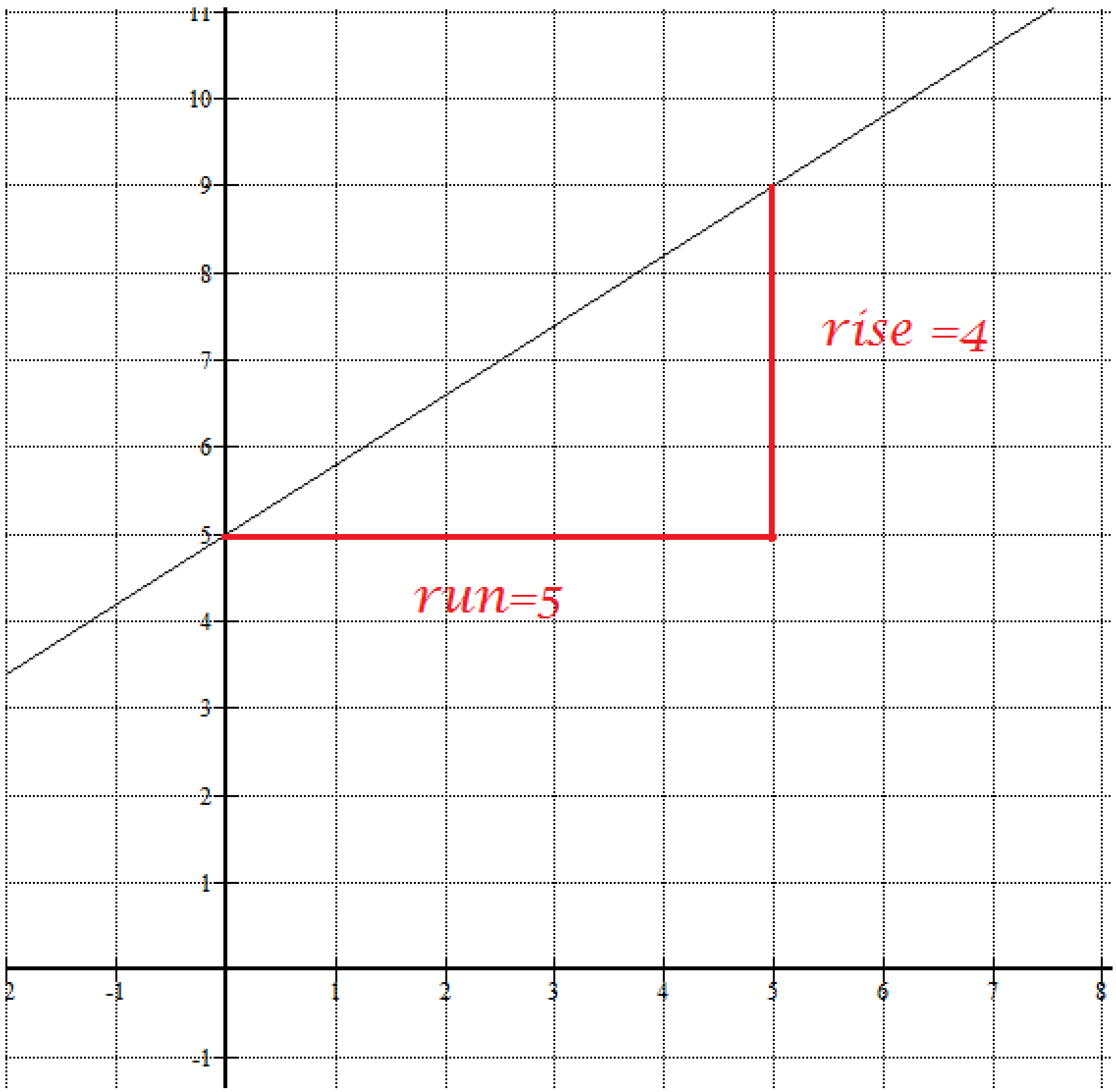
Case 2; given the gradient and a point

A straight line CD passes through the point $C(4,3)$ and has a gradient of $m = \frac{3}{4}$, calculate the equation of CD

Again using the point given and the general equation of the line we have the equation of CD as

$$y - 4 = \frac{3}{4}(x - 4)$$
$$y - 4 = \frac{3x}{4} - 3$$
$$y = \frac{3x}{4} - 3 + 4$$
$$y = \frac{3x}{4} + 1$$

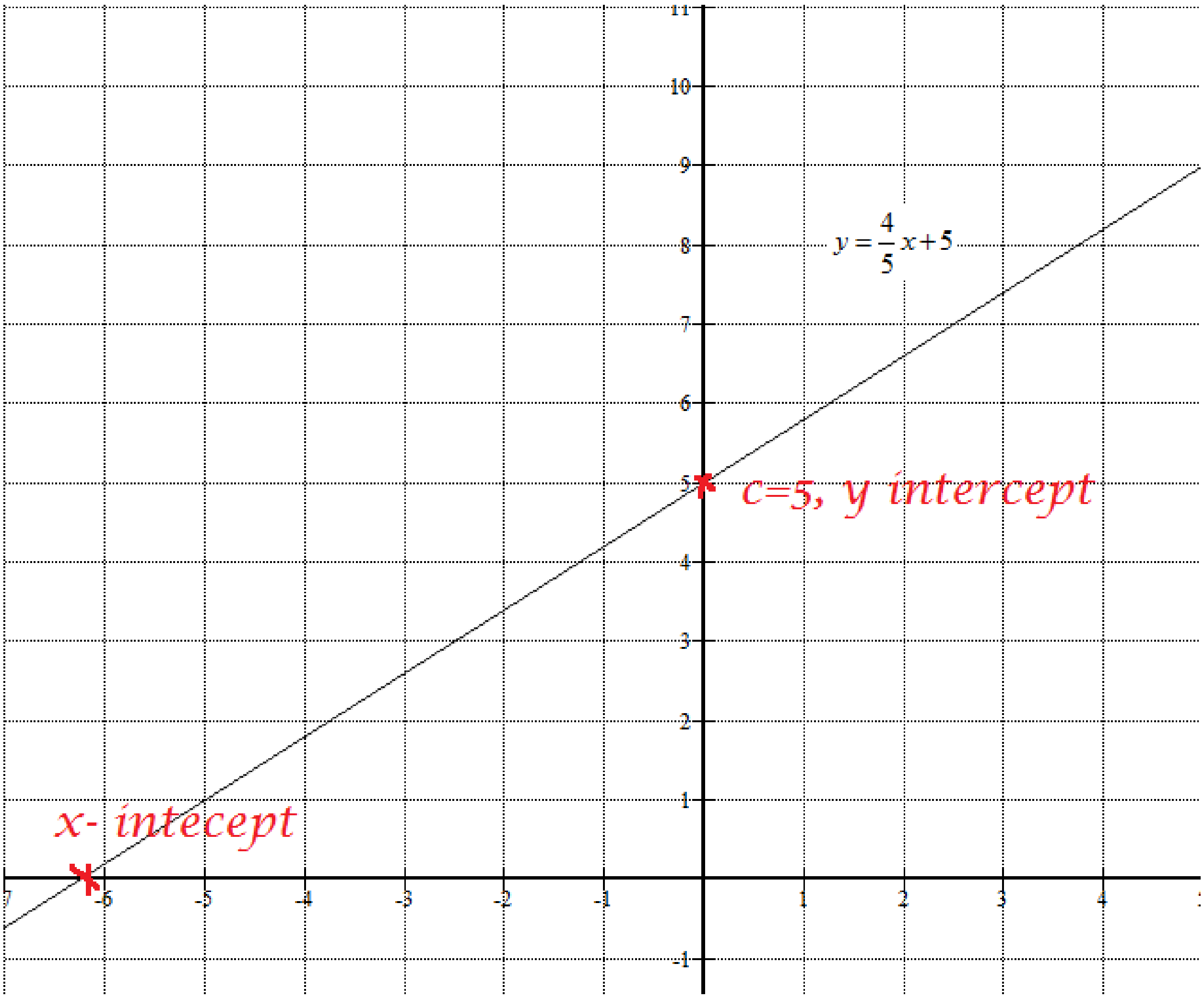
Case 3; find the equation of a line given its graph



Here we see that the gradient $m = \frac{rise}{run} = \frac{4}{5}$ and our $\boldsymbol{y}$ intercept is 5, using the slope intercept of the line $y = mx + c$ we have by substitution the equation of the line $y = \frac{4}{5}x + 5$

Interpreting the x and y intercepts of a straight line

Consider the graph below



We already understand that the y – intercept is the point at which the line cute/intersects the y – axis

The x – intercept is the solution of the equation $\frac{4}{5}x+5=0$ which gives

$$\frac{4}{5}x=-5$$

$$x=\frac{-25}{4}=-6\frac{1}{4}$$

Practice Questions

1. <u>Jun 85</u>
   A quadrilateral ABCD is formed by joining the points whose coordinates are A(–2, o), B(0, 4), C(7, 3), and D(3, –5)
   a. Calculate the length of AC
   b. Show that BD is perpendicular to AC
   c. Prove that ABCD is a trapezium.

2. <u>Jun 88</u>
   The coordinates of A and B are (3, 5) and (7, 1) respectively. X is the midpoint of AB.
   a. Calculate
      i. the length of AB
      ii. the gradient of AB
      iii. the coordinates of X.
   b. Determine the equation of the perpendicular bisector of AB and state

      the coordinates of the point at which the perpendicular bisector meets the *y*–axis

3. <u>Jan 90</u>
   A straight line HK cuts the *y*–axis at (0 –1). The gradient of HK is $\frac{2}{3}$.
   Show that the equation of the line HK is $2x - 3y = 3$.

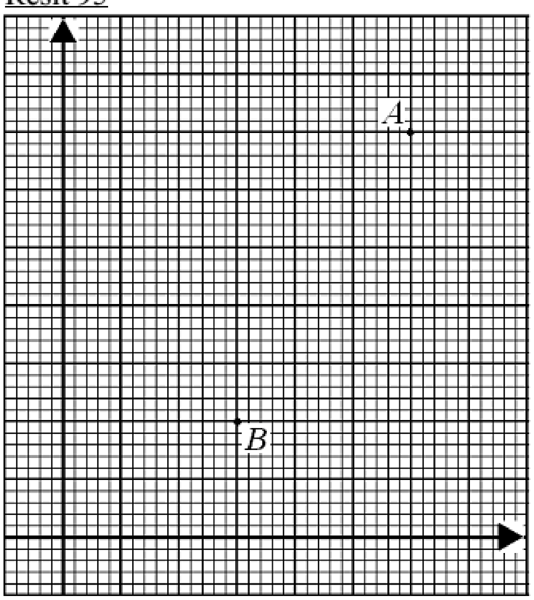
4. <u>Jun 94</u>
   The coordinates of the points A and B are $(5, 24)$ and (–10, –12) respectively.
   a. Calculate the gradient of the line joining A and B

b. Determine the equation of AB.
c. State the coordinates of the y–axis intercept for the line AB.

5. Jun 95
A straight line is drawn through the points $A(-5, 3)$ and $B(1, 2)$
a. Determine the gradient of AB
b. Write the equation of the line AB

6. Jan 96
The coordinates of A and B are $(3, 1)$ and $(-1, 3)$ respectively.
i. Find the gradient of the line AB.
ii. State the coordinates of the midpoint of A and B
iii. Hence determine the equation of the perpendicular bisector of AB.

7. Resit 95



The diagram above shows the two points A(6, 7) and B (3, 2)

a. Calculate the gradient of AB
b. Determine the equation of the line AB
c. Obtain the value of x, if a point P(x, -6) lies on AB

8. Jan 98

The equation of a line ,L, is $5x - 2y = 9$

i. Write the equation of L in the form $y = mx + c$

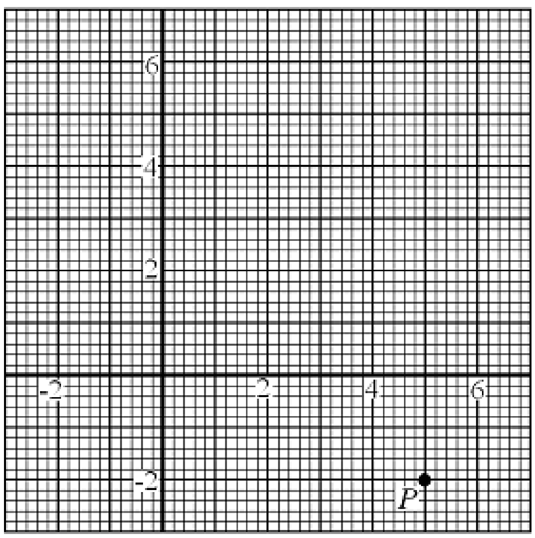
ii. Hence, state the gradient of the line L

iii. A point, N, with coordinates (h, h) lies on the line. Calculate the value of h.

iv. Find the equation of the line through (0, 2) perpendicular to L.

9. Jun 98

on the graph below, the point P(x, y) has been marked in



i. Write down the coordinates of ***P***.

ii. Through ***P***, draw a straight line whose $y-$axis intercept is 4.

iii. Calculate the gradient of the straight line.

iv. Determine the equation of the straight line