STRAND A: Computation

Unit 3 *Using Fractions and Percentages*

Student Text

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3.1 Addition and Subtraction of Fractions

Note

The *numerator* is the **top** part of a fraction and the *denominator* is the **bottom** part of a fraction.

When adding or subtracting fractions they must have the same *denominator*.

Worked Example 1

$$\frac{4}{7} + \frac{5}{7} = ?$$

Solution

As both fractions have the same denominator (7), they can simply be added to give

$$\frac{4}{7} + \frac{5}{7} = \frac{9}{7}$$

$$= 1\frac{2}{7}$$

Worked Example 2

$$\frac{3}{4} + \frac{2}{5} = ?$$

Solution

As these fractions have different denominators, it is necessary to find the *lowest common multiple* of the denominator, that is, the smallest number into which both denominators will divide exactly (we sometimes refer to this as the *lowest common denominator*). In this case it is 20, since both 4 and 5 divide into 20 exactly.

$$\frac{3}{4} + \frac{2}{5} = \frac{15}{20} + \frac{8}{20}$$

$$= \frac{15 + 8}{20}$$

$$= \frac{23}{20}$$

$$= 1\frac{3}{20}$$

Worked Example 3

$$\frac{2}{3} + \frac{7}{12} = \ ?$$

Solution

In this example, 12 is the lowest common denominator.

$$\begin{aligned} \frac{2}{3} + \frac{7}{12} &= \frac{8}{12} + \frac{7}{12} \\ &= \frac{8+7}{12} \\ &= \frac{15}{12} \\ &= 1\frac{\cancel{3}^{\,1}}{\cancel{12}_{\,4}} \\ &= 1\frac{1}{4} \end{aligned}$$

Worked Example 4

$$\frac{5}{8} - \frac{1}{3} = \ ?$$

Solution

Here 24 is the lowest common denominator.

$$\begin{aligned} \frac{5}{8} - \frac{1}{3} &= \frac{15}{24} - \frac{8}{24} \\ &= \frac{15-8}{24} \\ &= \frac{7}{24} \end{aligned}$$

Exercises

1. Give the answers to the following, simplifying them as far as possible.

(a) $\frac{1}{5} + \frac{1}{5}$ (b) $\frac{3}{8} + \frac{1}{8}$ (c) $\frac{5}{7} + \frac{1}{7}$

(d) $\frac{5}{7} - \frac{2}{7}$ (e) $\frac{8}{13} - \frac{5}{13}$ (f) $\frac{7}{9} - \frac{4}{9}$

(g) $\frac{7}{9} + \frac{8}{9}$ (h) $\frac{3}{5} + \frac{4}{5}$ (i) $\frac{6}{7} + \frac{5}{7}$

(j) $\frac{7}{10} - \frac{3}{10}$ (k) $\frac{8}{9} - \frac{5}{9}$ (l) $\frac{4}{15} - \frac{1}{15}$

2. Complete each of the following.

(a) $\frac{2}{5} + \frac{3}{7} = \frac{?}{35} + \frac{15}{35}$

$= \frac{?}{35}$

(b) $\frac{1}{5} + \frac{1}{6} = \frac{?}{30} + \frac{?}{30}$

$= \frac{?}{30}$

(c) $\frac{1}{2} + \frac{1}{4} = \frac{?}{4} + \frac{1}{4}$

$= \frac{?}{4}$

(d) $\frac{3}{16} + \frac{5}{8} = \frac{3}{16} + \frac{?}{16}$

$= \frac{?}{16}$

(e) $\frac{4}{7} + \frac{2}{3} = \frac{?}{21} + \frac{?}{21}$

$= \frac{?}{21}$

(f) $\frac{3}{5} + \frac{7}{12} = \frac{?}{60} + \frac{?}{60}$

$= \frac{?}{60}$

3. Find the answers to the following, simplifying them if possible.

(a) $\frac{1}{6} + \frac{3}{8}$ (b) $\frac{5}{7} + \frac{2}{5}$ (c) $\frac{1}{8} + \frac{3}{32}$

(d) $\frac{1}{10} + \frac{1}{3}$ (e) $\frac{3}{7} + \frac{5}{8}$ (f) $\frac{1}{2} + \frac{2}{3}$

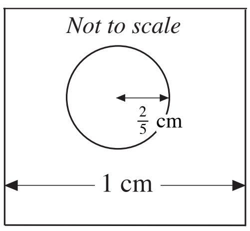
(g) $\frac{1}{7} + \frac{1}{10}$ (h) $\frac{5}{8} + \frac{4}{3}$ (i) $\frac{6}{7} + \frac{2}{3}$

(j) $\frac{4}{7} - \frac{1}{2}$ (k) $\frac{6}{11} - \frac{1}{4}$ (l) $\frac{2}{3} - \frac{1}{6}$

(m) $\frac{3}{4} - \frac{2}{3}$ (n) $\frac{5}{8} - \frac{5}{12}$ (o) $\frac{11}{12} - \frac{3}{8}$

4. A garden has an area of $\frac{2}{5}$ acre. The owner buys an extra $\frac{1}{3}$ acre of land to increase the size of the garden. What is the new size of the garden?

5. A company makes a profit of $\$\frac{3}{4}$ million in one year and $\$\frac{2}{3}$ million the next year. Find the total profits for the two-year period.

6. 

A hole of radius $\frac{2}{5}$ cm is drilled in the middle of a metal sheet of width 1 cm.

How far is it from the edge of the sheet to the hole?

7. A council decides to turn $\frac{1}{3}$ of a park into a dog-free zone. It later bans dogs from the play area which occupies $\frac{1}{10}$ of the park and which was originally outside the dog-free zone. What fraction of the park is now open to dogs?

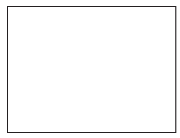
8. Mike has filled $\frac{3}{5}$ of the space on the hard disc in his computer with software. He wants to keep $\frac{1}{4}$ of the disc free from software. What fraction of the disc is left for extra software?

9. In a school $\frac{1}{3}$ of the students walk to school, $\frac{1}{2}$ travel by route taxi and the rest by bus. What fraction of the children travel by bus?

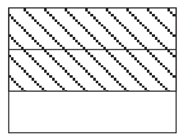
10. A shopper buys $1\frac{1}{4}$ kg of cabbage and $1\frac{1}{3}$ kg of onions. Find the total weight of vegetables bought.

3.2 Multiplication and Division of Fractions

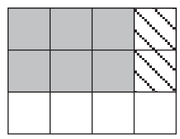
Multiplication



Consider finding $\frac{3}{4}$ of $\frac{2}{3}$ by starting with this rectangle.



First select $\frac{2}{3}$ of the rectangle, as shown by the hatched area.



Then select $\frac{3}{4}$ of the hatched area.

This area represents $\frac{3}{4}$ of $\frac{2}{3}$ of the original rectangle,

that is, $\frac{6}{12}$ or $\frac{1}{2}$ of the original rectangle.

$\frac{3}{4}$ of $\frac{2}{3}$ is the same as $\frac{3}{4} \times \frac{2}{3}$, so

$$\frac{3}{4} \times \frac{2}{3} = \frac{6}{12} = \frac{1}{2}$$

When multiplying two fractions, the *numerators* (top parts) should be multiplied together to give the numerator of the result. Similarly, the two denominators should be multiplied together.

In general terms,

$$\boxed{\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}}$$

Worked Example 1

$$\frac{3}{4} \times \frac{5}{7} = ?$$

Solution

$$\frac{3}{4} \times \frac{5}{7} = \frac{3 \times 5}{4 \times 7}$$

$$= \frac{15}{28}$$

Worked Example 2

$$\frac{3}{5} \times \frac{7}{12} = ?$$

Solution

$$\frac{\overset{1}{\cancel{3}}}{5} \times \frac{7}{\cancel{12}_4} = \frac{1 \times 7}{5 \times 4}$$

$$= \frac{7}{20}$$

Worked Example 3

$$1\frac{1}{2} \times 3\frac{4}{5} = ?$$

Solution

$$1\frac{1}{2} \times 3\frac{4}{5} = \frac{3}{2} \times \frac{19}{5}$$

$$= \frac{57}{10}$$

$$= 5\frac{7}{10}$$

Worked Example 4

Calculate the exact value of

$$\left(3\frac{3}{5} \times 1\frac{2}{3}\right) - 2\frac{2}{7}$$

Solution

$$3\frac{3}{5} \times 1\frac{2}{3} = \frac{18}{5} \times \frac{5}{3} = 6$$

so

$$\left(3\frac{3}{5} \times 1\frac{2}{3}\right) - 2\frac{2}{7} = 6 - 2\frac{2}{7}$$

$$= 3\frac{5}{7} \quad \left(\text{or } \frac{26}{7}\right)$$

Challenge!

The sum of $\frac{1}{2}$, $\frac{1}{3}$ *and* $\frac{1}{4}$ *of the enrolment of School A is exactly the enrolment of School B.*

The sum of $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$ *and* $\frac{1}{8}$ *of the enrolment of School A is exactly the enrolment of School C.*

What are the enrolments of these three schools, assuming that no school has more than 1000 students?

Division

To understand how to *divide* with fractions, first consider how multiplication and division are related.

Take as an example,

$$3 \times 4 = 12$$

Then it is also true that

$$12 \div 4 = 3$$

We say that '×4' and '÷4' are *inverse* (reversed or opposite) operations.

Note that

$$12 \times \frac{1}{4} = 3$$

so $\div 4$ is the same as $\times \frac{1}{4}$.

Similarly, because $\div \frac{1}{2}$ is the same as $\times 2$,

$$6 \div \frac{1}{2} = 12 \qquad \left(\text{check: } 12 \times \frac{1}{2} = 6\right)$$

and, alternatively, $6 \times 2 = 12$.

So $\div \frac{1}{2}$ is the same as $\times 2$.

You can generalise these examples to give

$$\div a \quad \text{is the same as} \quad \times \frac{1}{a}$$

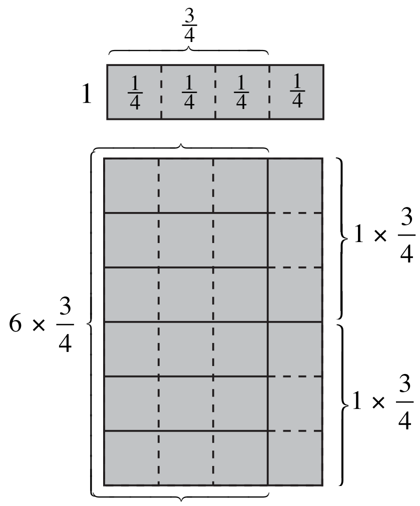
$$\div \frac{1}{b} \quad \text{is the same as} \quad \times b$$

and combining the two results gives

$$\div \frac{a}{b} \quad \text{is the same as} \quad \times \frac{b}{a}$$

For example,

$$6 \div \frac{3}{4} = 6 \times \frac{4}{3}$$
$$= 8$$



(This result, that there are 8 lots of $\frac{3}{4}$ in 6, is shown in the diagram opposite.)

Similarly,

$$\frac{6}{20} \div \frac{2}{5} = \frac{6}{20} \times \frac{5}{2}$$
$$= \frac{3}{4}$$

So to divide by a fraction, the fraction should be *inverted*, that is, turned upside down, and then multiplied.

In general terms,

$$\boxed{\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}}$$

Worked Example 5

$$\frac{3}{4} \div \frac{7}{8} = ?$$

Solution

$$\frac{3}{4} \div \frac{7}{8} = \frac{3}{\cancel{4}_1} \times \frac{\cancel{8}^2}{7}$$
$$= \frac{3 \times 2}{1 \times 7}$$
$$= \frac{6}{7}$$

Worked Example 6

$$\frac{\frac{1}{5} \times \frac{3}{10}}{2\frac{1}{2}} = ?$$

Solution

$$\frac{\frac{1}{5} \times \frac{3}{10}}{2\frac{1}{2}} = \frac{3}{50} \div \frac{5}{2}$$

$$= \frac{3}{50} \times \frac{2}{5}$$

$$= \frac{6}{250}$$

$$= \frac{3}{125}$$

Exercises

1. Find each of the following, cancelling when possible.

(a) $\frac{3}{4} \times \frac{5}{7}$ (b) $\frac{1}{5} \times \frac{7}{8}$ (c) $\frac{4}{5} \times \frac{1}{12}$

(d) $\frac{3}{7} \times \frac{9}{10}$ (e) $\frac{4}{7} \times \frac{5}{8}$ (f) $\frac{6}{7} \times \frac{3}{4}$

(g) $\frac{2}{7} \times \frac{3}{8}$ (h) $\frac{1}{6} \times \frac{4}{7}$ (i) $\frac{3}{5} \times \frac{10}{9}$

(j) $1\frac{1}{2} \times 1\frac{1}{3}$ (k) $4\frac{1}{6} \times 2\frac{1}{2}$ (l) $1\frac{3}{4} \times 2\frac{1}{7}$

(m) $3\frac{3}{7} \times 4\frac{1}{5}$ (n) $5\frac{1}{2} \times 1\frac{3}{4}$ (o) $8\frac{1}{2} \times 3\frac{4}{7}$

(p) $2\frac{3}{4} \times 4\frac{1}{7}$ (q) $5\frac{3}{8} \times 1\frac{5}{6}$ (r) $1\frac{2}{7} \times 1\frac{3}{8}$

2. Find

(a) $\frac{3}{4} \div \frac{1}{2}$ (b) $\frac{6}{7} \div \frac{3}{4}$ (c) $\frac{1}{5} \div \frac{1}{7}$

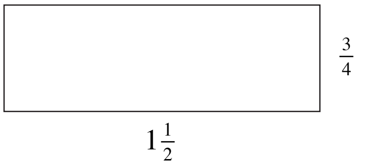
(d) $\frac{3}{8} \div \frac{4}{5}$ (e) $\frac{3}{7} \div \frac{9}{10}$ (f) $\frac{7}{4} \div \frac{2}{5}$

(g) $1\frac{1}{4} \div \frac{3}{4}$ (h) $5\frac{1}{2} \div \frac{1}{4}$ (i) $1\frac{1}{7} \div 2\frac{3}{8}$

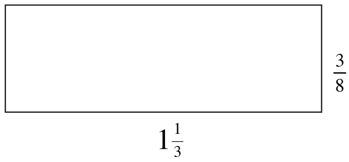
(j) $4\frac{1}{2} \div 1\frac{1}{5}$ (k) $1\frac{3}{4} \div 1\frac{5}{8}$ (l) $3\frac{1}{7} \div 1\frac{7}{8}$

3. Find the area of each rectangle below.

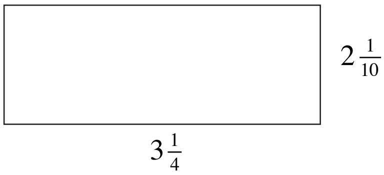
(a)



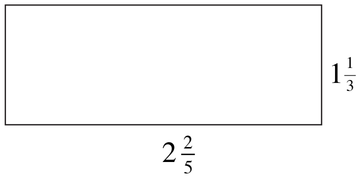
(b)



(c)



(d)

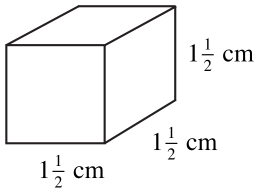


4. Mary has a garden. She grows vegetables on $\frac{1}{2}$ of her garden. On $\frac{1}{4}$ of this vegetable area she grows onions. What fraction of the garden is used for growing onions?

5. Using a calculator, or otherwise, determine the exact value of

$$\frac{3\frac{1}{3} - 2\frac{3}{5}}{2\frac{1}{5}}$$

6. A cube is made with sides of length $1\frac{1}{2}$ cm.



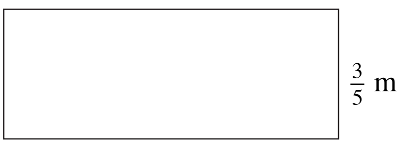
Find the volume and surface area of the cube.

7. A water can holds $5\frac{1}{2}$ litres when full. How much water is in the can if it is $\frac{3}{4}$ full?

8. Calculate

$$1\frac{3}{4} \div \left(2\frac{1}{2} - 1\frac{1}{3}\right)$$

9. Find the length of the unmarked side of this rectangle if its area is $1\frac{1}{2}\ \text{m}^2$.



10. A recipe requires $\frac{1}{4}$ kg of sugar for a cake. How many cakes could be made with $1\frac{3}{4}$ kg of sugar?

11. Amy runs 3 miles in $\frac{2}{3}$ hour. What is her speed?

12. It takes a factory $\frac{3}{4}$ hour to assemble a finished product. How many items could be assembled in an 8 hour day?

3.3 Compound Interest and Depreciation

Simple interest is where interest is paid at the end of the first year of an investment, that is, on the initial amount invested (the principal). The interest, though, is not re-invested and so the principal remains the same and, if interest rates do not change, the same amount of interest is paid again at the end of the next year.

Compound interest is when the interest paid each year is not paid out but is added to the principal. So the value of the account increases each year and the amount of interest earned each year will also increase. We say that the interest is *compounded*.

Worked Example 1

A person invests \$200 in a building society account which pays 4% interest at the end of each year.

Find the value of the investment after 3 years if the interest is compounded.

Solution

Interest of 4% will be added at the end of each year by multiplying by 1.04.

So,	value of account after 1 year:	\$200 × 1.04	= \$208
	value of account after 2 years:	\$208 × 1.04	= \$216.32
	value of account after 3 years:	\$216.32 × 1.04	= \$224.97

Note that the amount of interest added increases each year.

The final value could have been found in one calculation:

$$\$200 \times 1.04^3 = \$224.97$$

Worked Example 2

When Gemma was born, her grandmother invested \$200 in a building society for her. Find the value of this investment after 18 years if the interest rate is 6% per year and the interest is compounded.

Solution

$$\text{Final value} = \$200 \times 1.06^{18}$$
$$= \$570.87$$

Problems with *depreciation* can be tackled in a similar way.

Worked Example 3

A boat is bought for $14 000. Its value decreases by 8% each year. Find the value of the boat after:

(a) 1 year (b) 5 years (c) 10 years.

Solution

Decreasing the value by 8% leaves 92% of the original value.

(a) Value after one year $= \$14\,000 \times 0.92$
$= \$12\,880$

(b) Value after 5 years $= \$14\,000 \times 0.92^5$
$= \$9227.14$

(c) Value after 10 years $= \$14\,000 \times 0.92^{10}$
$= \$6081.44$

Worked Example 4

A music system has a cash price of $14 000. Under a hire purchase arrangement a 20% deposit is required, plus monthly instalments of $650 for two years.

How much money is saved by paying cash?

Solution

$$\text{Deposit} = \frac{20}{100} \times \$14\,000$$
$$= \$2800$$

$$\text{Total instalments} = 24 \times \$650$$
$$= \$15\,600$$

$$\text{Total hire purchase price} = \$2800 + \$15\,600$$
$$= \$18\,400$$

$$\text{Amount saved by paying cash} = \$18\,400 - \$14\,000$$
$$= \$4400$$

Note

You can see from these worked examples that the total amount in an account after *n* years, A_n, with interest of r % is given by

$$A_n = \left(1 + \frac{r}{100}\right)^n A_0$$

where A_0 is the initial sum invested.

Exercises

1. Jane invests $1200 in a bank account which earns compound interest at the rate of 6% per annum. Find the value of her investment after:

   (a) 1 year (b) 2 years (c) 5 years.

2. A sum of $5000 is to be invested for 10 years. What is the final value of the investment if the annual compound interest rate is:

   (a) 5% (b) 4.8% (c) 7.2%?

3. Which of the following investments would earn most interest?

   A $300 for 5 years at 2% compound interest per annum,

   B $500 for 1 year at 3% compound interest per annum,

   C $200 for 3 years at 8% compound interest per annum

4. The value of a computer depreciates at a rate of 25% per annum. In the USA, a new computer costs $1600. What will the value of the computer be after:

   (a) 2 years (b) 6 years (c) 10 years?

5. A car costs $9000 and depreciates at a rate of 20% per annum. Find the value of the car after 3 years.

6. Mr Mitchell deposited $40 000 in a bank and was paid *simple* interest at 7% per annum for two years.

   (a) Calculate the amount of interest he will have received in total by the end of the two-year period.

   (b) Mr Williams bought a plot of land for $40 000. The value of the land appreciated by 7% each year.

   Calculate the value of the land after a period of two years.

7. If the rate of inflation were to remain constant at 3%, find what the price of a pack of batteries, currently priced at J$158, would be in 4 years' time.

8. The population of a third world country is 42 million and growing at 2.5% per annum.
   (a) What size will the population be in 3 years' time?
   (b) In how many years' time will the population exceed 50 million?

9. The value of a boat depreciates at 15% per annum. A man keeps a boat for 4 years and then sells it.
   (a) If the boat initially cost $6000, find:
       (i) its value after 4 years,
       (ii) the selling price as a percentage of the original value.
   (b) Repeat (a) for a boat which cost $12 000.
   (c) Comment on your answers.

10. A couple borrow J$100 000 to furnish their new home. They have to pay interest of 18% on this amount.
   (a) Find the amount of interest which would be charged at the end of the first year.
   (b) If they repay J$30 000 at the end of each year, how much do they owe at the end of the third year of the loan?

Information

In 1996 a Japanese mathematician (using a computer!) took just 5 days to compute the value of π *to over 6 billion digits.*

3.4 Reverse Percentage Problems

Sometimes it is necessary to *reverse* percentage problems. For example, if the price of a television set includes GCT, you might need to know how much of the price is the GCT.

Worked Example 1

The price of a computer is $1398, including GCT at $16\frac{1}{2}\%$. Find the actual cost of the computer and the amount of GCT which has to be paid.

Solution

To add 16.5% GCT to a price it should be *multiplied* by 1.165. So to remove the GCT the price should be *divided* by 1.165.

$$\text{Original Price} = \frac{\$1398}{1.165} = \$1200$$

$$\text{GCT} = \$1398 - \$1200 = \$198$$

Worked Example 2

A customer is offered a 20% discount when buying a new bed. The discounted price is \$158.40. Find the full price of the bed.

Solution

To find the discounted price of the bed, the full price should be *multiplied* by 0.8.
So to find the full price, the discounted price should be *divided* by 0.8.

$$\text{Full price} = \frac{\$158.40}{0.8}$$
$$= \$198$$

Worked Example 3

Sharon invests some money in a building society at 6% interest per annum. After two years the value of her investment is J\$28 090. Find the amount she invested.

Solution

To find the final value, the amount invested would be *multiplied* by 1.06^2.
To find the amount invested, *divide* the final value by 1.06^2.

$$\text{Amount invested} = \frac{\text{J\$28 090}}{1.06^2}$$
$$= \text{J\$25 000}$$

Exercises

1. A tourist in the USA can reclaim the VAT he has paid on the following items, the prices of which include VAT.

Camera	\$149.60
CD player	\$110.45
Watch	\$42.77
DVD player	\$406.08

   (a) Find the total cost of the items without VAT at 16.5%.
   (b) How much VAT can the tourist reclaim?

2. The price of a television set is J\$22 560 including 17.5% VAT. What would be the price with no VAT?

3. A gas bill of J\$4345 includes VAT at 8%. Find the amount of VAT paid.

4. The end of year profits of a company increased this year by 12% to \$90 944. Find the profits made last year.

5. A special bottle of dishwashing liquid contains 715 ml of liquid. The bottle is marked '30% extra free'. How much liquid is there in a normal bottle?

6. In a sale the following items are offered at discount prices as listed.

Item	*Sale Price*	*Discount*
DVD player	\$288.00	10%
TV set	\$373.12	12%
Computer	\$1124.80	24%
Calculator	\$13.78	5%

What were the prices of these items before the sale?

7. After one year, the value of a car has fallen by 15% to \$8330. What was the value of the car at the beginning of the year?

8. A sum is invested in a building society at 4% interest per annum and after 3 years the value of the investment is \$562.43. How much was originally invested?

9. Jenny's pocket money is increased by 25% each year on her birthday. When she is 16 years old, her pocket money is J\$1286 per week. How much did she get per week when she was:

   (a) 15 years old (b) 13 years old (c) 10 years old?

10. Jai buys a car, keeps it for 4 years and then sells it for \$2100. If the value of the car has depreciated by 12% per year, how much did Jai originally pay for the car?

Information

The Chinese represented negative numbers by indicating them in red and the Hindus denoted them by putting a circle or a dot over the numbers. The Chinese had knowledge of negative numbers as early as 200 BC and the Hindus as early as the 7th century.

In Europe, as late as the 16th century, some scholars still regarded negative numbers as absurd. In 1545, Cardano (1501–1570), an Italian scholar, called positive numbers 'true' and negative numbers 'fictitious' numbers.