

# STRAND H: Relations, Functions and Graphs

## Unit 30    *Functions*

### Student Text

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# 30 Functions

## 30.1 Functions, Mappings and Domains

You are already familiar with the idea of a function. Here we make the concept more formal with a number of definitions that we will introduce. We illustrate the definitions using examples, including the formula that relates pressure to volume:

$$p = \frac{k}{v}$$

Here  $p$  is the pressure of a gas,  $v$  is its volume and  $k$  is a constant.



### Worked Example 1

If  $p = \frac{10}{v}$ , use the mapping diagram opposite to show how  $v$  maps to  $p$  for  $1 \leq v \leq 10$ .



### Solution

Clearly if

$$v = 1, \quad p = \frac{10}{1} = 10$$

$$v = 2, \quad p = \frac{10}{2} = 5$$

$$v = 5, \quad p = \frac{10}{5} = 2$$

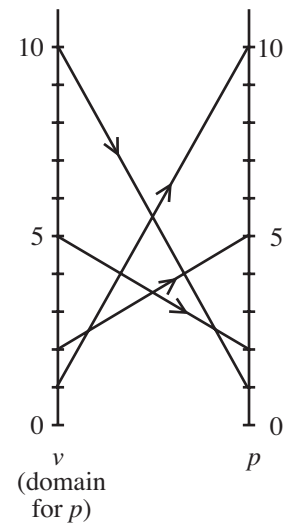
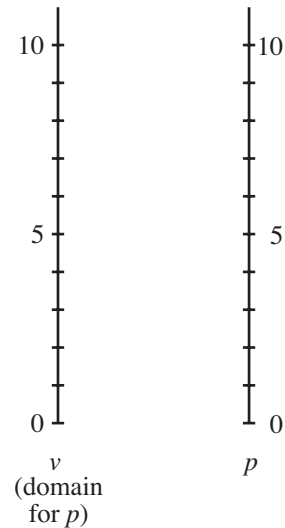
$$v = 10, \quad p = \frac{10}{10} = 1$$

We can illustrate this mapping.

In fact, for any value  $1 \leq v \leq 10$ , we can map  $v$  to  $p$ .

For example,

$$v = 4, \quad p = \frac{10}{4} = \frac{5}{2}, \text{ etc.}$$





## Notes

1. The set of values of  $v$ , defined here as  $1 \leq v \leq 10$ , is called the **DOMAIN** of the mapping.
2. The set of values of  $p$  that  $v$  maps onto is known as the **RANGE**; here the range is also  $1 \leq p \leq 10$ .
3. For every value of  $v$  in its domain, there is a unique value of  $p$ . This is called a 1 : 1 mapping



## Worked Example 2

For the function  $y = f(x) = x^2$ ,  $-5 \leq x \leq 5$ , complete a similar mapping diagram.



## Solution

Here, notice that

$$f(-5) = (-5)^2 = 25$$

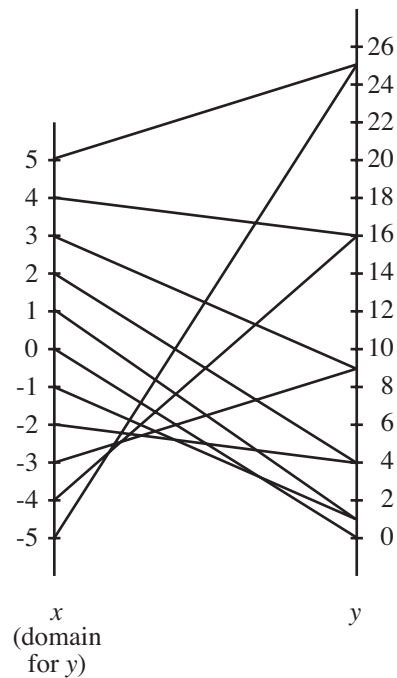
$$f(-2) = (-2)^2 = 4$$

$$f(0) = 0$$

$$f(5) = 25$$

etc.

We can complete the mapping opposite for some of the values.



## Notes

1. In this example, apart from 0 (which maps to 0), there are two values which both map to the same value. For example, +2 and -2 both map to 4. This is **NOT** a 1 : 1 mapping.
2. As well as the notation  $y = f(x)$  for a function, we can also use the mapping notation

$$f : x \rightarrow x^2$$



### Worked Example 3

If  $f$  is defined by

$$f: x \rightarrow x^3 - x$$

what are the values of

- (a)  $f(-1)$       (b)  $f(0)$       (c)  $f(1)$       (d)  $f(5)$  ?



### Solution

(a)  $f(-1) = (-1)^3 - (-1) = -1 + 1 = 0$

(b)  $f(0) = 0^3 - 0 = 0$

(c)  $f(1) = 1^3 - 1 = 1 - 1 = 0$

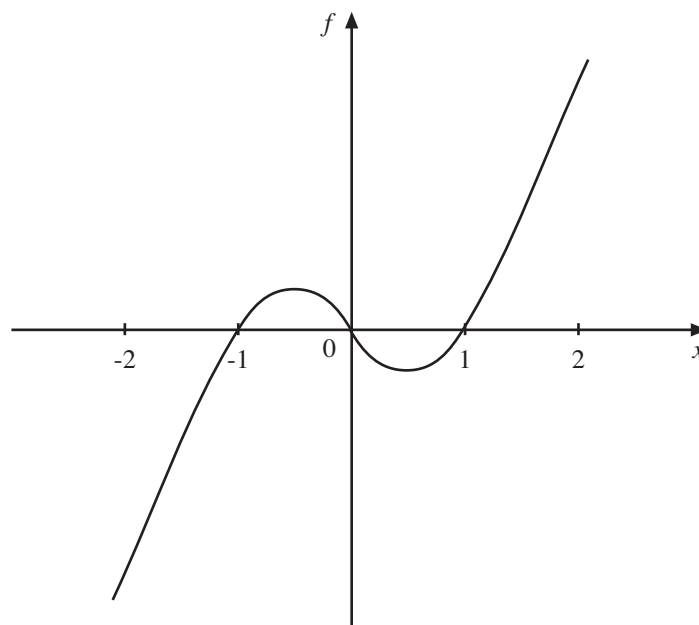
(d)  $f(5) = 5^3 - 5 = 125 - 5 = 120$



### Note

This is not a 1 : 1 mapping as  $f(-1) = f(0) = f(1)$ .

This can be seen in a sketch of the function, as below.





## Exercises

1.  $f : x \rightarrow 2x - 1, 1 \leq x \leq 6$
- Complete a mapping diagram for integer values in the domain.
  - What is the range of this function?
  - Is this a 1 : 1 mapping?

2.  $f : x \rightarrow \frac{1}{2}x^2 - 1, -3 \leq x \leq 3$
- Complete a mapping diagram for integer values in the domain.
  - What is the range?
  - Is this a 1 : 1 mapping?

3. (a) If  $f$  is defined by

$$f : x \rightarrow \frac{x + 1}{x - 1} \quad (x \neq 1)$$

what are the values of

- $f(-2)$
- $f(-1)$
- $f(0)$
- $f(2)$
- $f(3)$  ?

- (b) Why is the domain restricted by  $x \neq 1$  ?

4. Given that  $h(x) = \frac{x^2 - 16}{x - 2}$ ,

calculate

- $h(-2)$
- the values of  $x$  for which  $h(x) = 0$ .

(CXC)

5.  $f : x \rightarrow \frac{3 - x}{1 - x}$

- What is the value of
  - $f(-2)$
  - $f(0)$
  - $f(2)$  ?
- Find the value of  $x$  for which  $f(x) = 4$ .

6.  $f$  is a function defined by the equation

$$f : x \rightarrow x^2 + 2$$

Find the value of

- (a)  $f(2)$       (b)  $f(-1)$       (c)  $f(0)$   
 (d)  $f(a^2)$       (e)  $f(1 - a)$

(where  $a$  is a constant real number).

(CXC)

7.  $g$  is defined by the equation

$$g : x \rightarrow \frac{1}{x} \quad (x \neq 0)$$

Find the value of

- (a)  $g(1)$       (b)  $g(-1)$       (c)  $g(0.01)$   
 (d)  $g(a^2)$       (e)  $g(1 - a)$

(where  $a$  is a constant real number).

(CXC)

8.  $f : x \rightarrow 1 - x^2$

- (a) What are the values of

- (i)  $f(-2)$       (ii)  $f(-1)$       (iii)  $f(0)$   
 (iv)  $f(1)$       (v)  $f(2)$

- (b) Find the value of  $x$  for which  $f(x) = -8$ .

9. Given that  $m * 1 = m^2 - lm$ ,

- (a) evaluate  $5 * 3$

- (b) solve for  $g$  given that  $g * 4 = -3$ .

(CXC)

## 30.2 Composite Functions

The next concept to cover is that of the *composite function*, that is, a 'function of a function'.



### Worked Example 1

The two functions  $f$  and  $g$  are defined by

$$f : x \rightarrow 4x - 3$$

$$g : x \rightarrow 2x - 1$$

- (a) Find  $f(g(x))$  and  $g(f(x))$ ,  
 (b) What are the values of  $f(g(0))$  and  $g(f(0))$ ?  
 (c) Are there any solutions of  $f(g(x)) = g(f(x))$ ?



## Solution

$$\begin{aligned}
 \text{(a)} \quad f(g(x)) &= 4(g(x)) - 3 && \text{since, for example, } f(a) = 4a - 3, \text{ etc.} \\
 &= 4(2x - 1) - 3 \\
 &= 8x - 4 - 3 \\
 &= 8x - 7
 \end{aligned}$$

$$\begin{aligned}
 g(f(x)) &= 2(f(x)) - 1 \\
 &= 2(4x - 3) - 1 \\
 &= 8x - 6 - 1 \\
 &= 8x - 7
 \end{aligned}$$

$$\text{(b)} \quad f(g(0)) = 8 \times 0 - 7 = -7$$

$$g(f(0)) = 8 \times 0 - 7 = -7$$

(c) As  $f(g(x)) = g(f(x))$ , then this is satisfied by all values of  $x$ .



## Note

The result  $f(g(x)) = g(f(x))$  is not generally true, but depends on the values of the coefficients in the functions (see Question 6 in Exercises).



## Note

We usually write, for short,

$$f(g(x)) = fg(x)$$

and

$$g(f(x)) = gf(x)$$

but you must remember what this notation means.



## Worked Example 2

The functions  $f$  and  $g$  are defined by

$$f : x \rightarrow x^2 + 1$$

$$g : x \rightarrow x - 1$$

Will  $fg(x) = gf(x)$  for any value of  $x$ ?



## Solution

$$\begin{aligned}
 fg(x) &= f(g(x)) = (g(x))^2 + 1 \\
 &= (x - 1)^2 + 1 \\
 &= x^2 - 2x + 1 + 1 \\
 &= x^2 - 2x + 2
 \end{aligned}$$

$$\begin{aligned}
 gf(x) &= g(f(x)) = f(x) - 1 \\
 &= x^2 + 1 - 1 \\
 &= x^2
 \end{aligned}$$

If  $fg(x) = gf(x)$ , then

$$\begin{aligned}
 x^2 - 2x + 2 &= x^2 \\
 -2x + 2 &= 0 \\
 2x &= 2 \\
 x &= 1
 \end{aligned}$$

So  $fg(x) = gf(x)$  when  $x = 1$ .



## Exercises

1. The functions  $f$  and  $g$  are defined as

$$f : x \rightarrow \frac{2x - 1}{x + 3} \qquad g : x \rightarrow 2x - 1$$

What is the value of

- $g(2)$
  - $f(-2)$
  - $fg(2)$
  - $gf(-2)$  ?
2. If  $f(x) = x - 1$  and  $g(x) = x^3$ ,
- find
    - $fg(x)$
    - $gf(x)$
  - Does  $fg(x) = gf(x)$  have any solutions?



3. The functions  $f$  and  $g$  are defined by

$$f : x \rightarrow \frac{1}{x} \qquad g : x \rightarrow x + 1$$

What is

(a)  $fg(x)$                       (b)  $gf(x)$  ?

4. Find the composite function  $fg(x)$  and  $gf(x)$  if

$$f(x) = 1 + \frac{1}{x}, \quad g(x) = x^2$$

The function  $h(x) = gf(x) - fg(x)$ .

Determine  $h(1)$  and  $h(-1)$ .

5.  $f : x \rightarrow x + 3$                        $g : x \rightarrow x - 3$

What is

(a)  $fg(x)$                       (b)  $gf(x)$  ?

6. If

$$f(x) = ax + b, \quad g(x) = cx + d$$

where  $a, b, c, d$  are constants, show that  $fg(x) = gf(x)$  only when  $ad + b = cb + d$ .

## 30.3 Inverse Functions

You have probably already met the formula for changing temperatures in degrees Celsius, °C, to degrees Fahrenheit, °F. It is

$$F = \frac{9}{5}C + 32 \qquad (1)$$

You can regard this as a 1 : 1 mapping and it can be transformed to give  $C$  in terms of  $F$ , as follows:

$$F - 32 = \frac{9}{5}C \qquad (\text{taking } -32 \text{ from each side})$$

$$\frac{5}{9}(F - 32) = C \qquad (\text{multiplying by } \frac{5}{9})$$

That is,  $C = \frac{5}{9}(F - 32) \qquad (2)$

If we write equations (1) and (2) in our functions notation, writing  $C$  for  $x$  in (1) and  $F$  for  $x$  in (2),

we have

$$f : x \rightarrow \frac{9}{5}x + 32$$

$$g : x \rightarrow \frac{5}{9}(x - 32)$$



### Worked Example 1

Show that  $fg(x) = gf(x) = x$  for the functions above.



### Solution

$$\begin{aligned} fg(x) &= f(g(x)) = \frac{9}{5}g(x) + 32 \\ &= \frac{9}{5} \left\{ \frac{5}{9}(x - 32) \right\} + 32 \\ &= x - 32 + 32 \\ &= x \end{aligned}$$

and

$$\begin{aligned} gf(x) &= g(f(x)) = \frac{5}{9}(f(x) - 32) \\ &= \frac{5}{9} \left( \frac{9}{5}x + 32 - 32 \right) \\ &= \frac{5}{9} - \frac{9}{5}x \\ &= x \end{aligned}$$



### Note

If functions  $f$  and  $g$  are such that

$$fg(x) = x = gf(x)$$

we say that  $g$  is the *inverse* of  $f$  and denote this by

$$f^{-1}(x) = g(x) \text{ or } f^{-1} = g$$

Similarly,  $f$  is the *inverse* of  $g$ , so

$$g^{-1}(x) = f(x) \text{ or } g^{-1} = f$$

We can see how to find inverse functions in the next Worked Example.



## Worked Example 2

If  $f(x) = \frac{1}{1-x} + 2$  ( $x \neq 1$ ), find its inverse function and state its domain.



### Solution

We write  $y = \frac{1}{1-x} + 2$  and, as with the temperature conversion above, use algebraic manipulation to write  $x$  as a function of  $y$ .

Starting with  $y = \frac{1}{1-x} + 2$ , take  $-2$  from each side to give

$$\begin{aligned} y - 2 &= \frac{1}{1-x} + 2 - 2 \\ &= \frac{1}{1-x} \end{aligned}$$

Multiply both sides by  $(1-x)$  to give

$$\begin{aligned} (1-x)(y-2) &= (1-x) \times \frac{1}{(1-x)} \\ (1-x)(y-2) &= 1 \end{aligned}$$

Now divide both sides by  $(y-2)$  to give

$$\begin{aligned} \frac{(1-x)(y-2)}{(y-2)} &= \frac{1}{(y-2)} \\ 1-x &= \frac{1}{y-2} \end{aligned}$$

or, multiplying throughout by  $-1$ ,

$$x-1 = \frac{1}{2-y}$$

and adding 1 to both sides, gives

$$x-1+1 = \frac{1}{2-y} + 1$$

or

$$x = 1 + \frac{1}{2-y}$$

This is the inverse function, which we could write as

$$f^{-1}(y): y \rightarrow 1 + \frac{1}{2-y}$$

Interchanging  $y$  and  $x$  (it is just a variable) gives

$$f^{-1}(x) = x \rightarrow 1 + \frac{1}{2-x}$$

or

$$f^{-1}(x) = 1 + \frac{1}{2-x}$$

The domain of  $f^{-1}(x)$  is  $x \neq 2$  (as the function is not defined at  $x = 2$ ).



## Note

1. We can check values of

$$f(x) = \frac{1}{1-x} + 2 \text{ and } f^{-1}(x) = 1 + \frac{1}{2-x}$$

For example,

$$f(2) = \frac{1}{1-2} + 2 = \frac{1}{-1} + 2 = -1 + 2 = 1$$

and 
$$f^{-1}(1) = 1 + \frac{1}{2-1} = 1 + \frac{1}{1} = 1 + 1 = 2$$

Similarly,

$$f(4) = \frac{1}{1-4} + 2 = -\frac{1}{3} + 2 = \frac{5}{3}$$

whilst

$$f^{-1}\left(\frac{5}{3}\right) = 1 + \frac{1}{\left(2 - \frac{5}{3}\right)} = 1 + \frac{1}{\left(\frac{1}{3}\right)} = 1 + 3 = 4$$

So we see that, for these values,

$$f^{-1}f(x) = x$$

2. In general,

$$\begin{aligned} f^{-1}f(x) &= f^{-1}(f(x)) = f^{-1}\left(\frac{1}{1-x} + 2\right), \text{ using the } f \text{ formula} \\ &= f^{-1}\left(\frac{1 + 2(1-x)}{(1-x)}\right) \\ &= f^{-1}\left(\frac{1 + 2 - 2x}{1-x}\right) \end{aligned}$$

$$\begin{aligned}
 &= f^{-1}\left(\frac{3-2x}{1-x}\right), \text{ and using the } f^{-1} \text{ formula} \\
 &= 1 + \frac{1}{2 - \left(\frac{3-2x}{1-x}\right)} \\
 &= 1 + \frac{1}{\frac{2(1-x) - (3-2x)}{1-x}} \\
 &= 1 + \frac{(1-x)}{(2-2x-3+2x)} \\
 &= 1 + \frac{(1-x)}{-1} \\
 &= 1 - 1 + x \\
 &= x
 \end{aligned}$$

So this proves that  $f$  and  $f^{-1}$  are inverse functions.



### Worked Example 3

Two functions,  $g$  and  $h$ , are defined as

$$g : x \rightarrow \frac{2x+3}{x-4} \text{ and}$$

$$h : x \rightarrow \frac{1}{x}.$$

Calculate

- the value of  $g(7)$
- the value of  $x$  for which  $g(x) = 6$ .

Write expressions for

- $hg(x)$
- $g^{-1}(x)$

(CXC)



## Solution

$$(a) \quad g(7) = \frac{2 \times 7 + 3}{7 - 4} = \frac{15}{3} = 5$$

$$(b) \quad 6 = \frac{2x + 3}{x - 4} \Rightarrow 6(x - 4) = 2x + 3$$

$$6x - 24 = 2x + 3$$

$$4x = 27$$

$$x = \frac{27}{4}$$

$$\text{Check} \quad g\left(\frac{27}{4}\right) = \frac{2 \times \frac{27}{4} + 3}{\frac{27}{4} - 4}$$

$$= \frac{54 + 12}{27 - 16}$$

$$= \frac{66}{11}$$

$$= 6$$

$$(c) \quad hg(x) = h(g(x))$$

$$= \frac{1}{g(x)}$$

$$= \frac{1}{\left(\frac{2x + 3}{x - 4}\right)}$$

$$= \frac{x - 4}{2x + 3}$$

(d) To find  $g^{-1}(x)$ , we write

$$y = \frac{2x + 3}{x - 4}$$

and find  $x$  as a function of  $y$ ; that is

$$y(x - 4) = 2x + 3$$

$$yx - 4y = 2x + 3$$

$$yx - 2x = 3 + 4y$$

$$x(y - 2) = 3 + 4y$$

$$x = \frac{3 + 4y}{y - 2} \Rightarrow g^{-1}(y) = \frac{3 + 4y}{y - 2}$$

So  $g^{-1}(x) = \frac{3 + 4x}{x - 2}$  (replacing  $y$  by  $x$ ).



## Exercises

1. Find the inverse function for each of these functions. In each case, state the domain of the inverse.

(a)  $f(x) = x + 2$

(b)  $f(x) = 4x - 2$

(c)  $f(x) = x$

(d)  $f(x) = \frac{3}{x}$  ( $x \neq 0$ )

(e)  $f(x) = \frac{1}{x + 2}$  ( $x \neq -2$ )

2. If  $f : x \rightarrow 4x - 3$ , find  $f^{-1}(x)$  and check that

$$f f^{-1}(x) = f^{-1} f(x) = x$$

3. The functions  $f$  and  $g$  are defined by

$$f(x) = \frac{1}{2}x + 5 \quad g(x) = x^2$$

Evaluate

(a)  $g(3) + g(-3)$

(b)  $f^{-1}(6)$

(c)  $f g(2)$

(CXC)

4. The functions  $f$  and  $g$  are defined as:

$$f(x) = \frac{2x - 1}{x + 3} \quad g(x) = 4x - 5$$

Determine:

(a)  $g(3)$

(b)  $f g(2)$

(c)  $f^{-1}(x)$

(CXC)

5. The function  $f$  is defined by

$$f : x \rightarrow \frac{1}{x} - 4$$

and has domain all  $x$  ( $x \neq 0$ ).

- (a) Find

(i)  $f\left(\frac{1}{4}\right)$

(ii)  $f(1)$

(iii)  $f^{-1}(0)$

- (b) Determine the inverse function  $f^{-1}(x)$  and use it to calculate

(i)  $f^{-1}(0)$

(ii)  $f^{-1}(-3)$

- (c) Show that  $f f^{-1}(x) = x$ .