STRAND E: Measurement

Unit 14 *Volumes*

Student Text

Contents

Section

denotes that the topic is not on the current CXC/CSEC Mathematics syllabus and therefore not examined, but is of relevance to the content of the Unit.

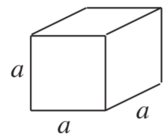
14 Volumes

14.1 Volumes of Cubes, Cuboids, Cylinders and Prisms

The volume of a *cube* is given by

$$V = a^3$$

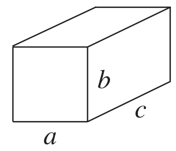
where a is the length of each side of the cube.



For a *cuboid* the volume is given by

$$V = abc$$

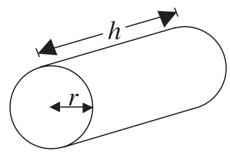
where a, b and c are the lengths shown in the diagram.



The volume of a *cylinder* is given by

$$V = \pi r^2 h$$

where r is the radius of the cylinder and h is its height.



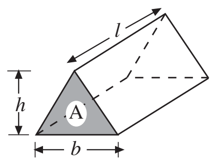
The volume of a *triangular prism* can be expressed in two ways, as

$$V = Al$$

where A is the area of the end and l the length of the prism, or as

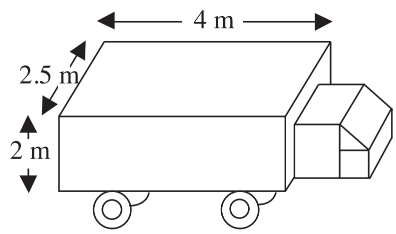
$$V = \frac{1}{2}bhl$$

where b is the base of the triangle and h is the height of the triangle.



Worked Example 1

The diagram shows a truck.



Find the volume of the load-carrying part of the truck.

Shading denotes that the topic is not on the current CXC/CSEC Mathematics syllabus and therefore not examined, but is of relevance to the content of the Unit.

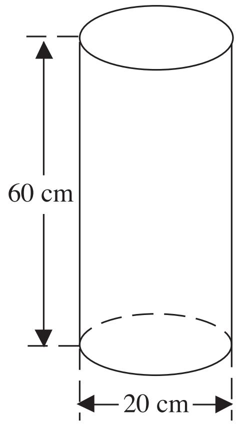
Solution

The load-carrying part of the truck is represented by a cuboid, so its volume is given by

$$V = 2 \times 2.5 \times 4$$
$$= 20 \text{ m}^3$$

Worked Example 2

The cylindrical body of a fire extinguisher has the dimensions shown in the diagram. Find the maximum volume of water the extinguisher could hold.



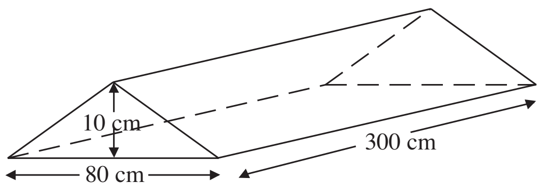
Solution

The body of the extinguisher is a cylinder with radius 10 cm and height 60 cm, so its volume is given by

$$V = \pi \times 10^2 \times 60$$
$$= 18850 \text{ cm}^3 \quad \text{(to the nearest cm}^3\text{)}$$

Worked Example 3

A traffic calming road hump (sleeping policeman) is made of concrete and has the dimensions shown in the diagram. Find the volume of concrete needed to make one road hump.



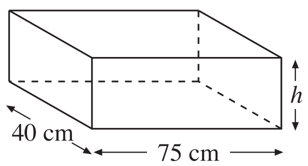
Solution

The shape is a triangular prism with $b = 80$, $h = 10$ and $l = 300$ cm. So its volume is given by

$$V = \frac{1}{2} \times 80 \times 10 \times 300$$
$$= 120\,000 \text{ cm}^3$$

Worked Example 4

The diagram below, **not drawn to scale**, shows a container in the shape of a rectangular prism.



The base of the container has a length of 75 cm and a width of 40 cm.

(i) Calculate the area, in cm^2, of the base of the container.

Water is poured into the container, reaching a height of 15 cm.

(ii) Calculate, in cm^3, the volume of water in the container.

(iii) If the container holds 84 litres when full, calculate the height, h, in cm, of the water when the container is full.

(CXC)

Solution

(i) Area of base $= 75 \times 40 = 3000 \text{ cm}^2$

(ii) Volume $= 15 \times 3000 \text{ cm}^2$

$= 45\,000 \text{ cm}^3$

(iii) When full, the tank holds $84 \times 1000 \text{ cm}^3$ of water, so

$$h \times 3000 = 84\,000$$

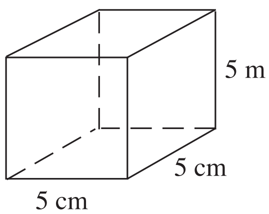
$$h = \frac{84\,000}{3000}$$

$$= 28 \text{ cm}$$

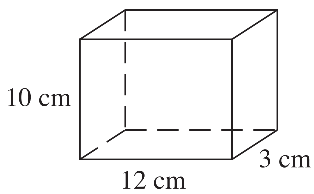
Exercises

1. Find the volume of each solid shown below.

(a)



(b)



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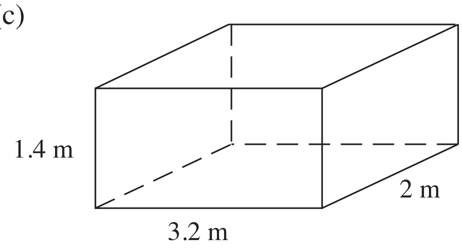
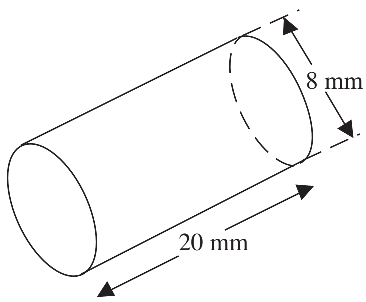
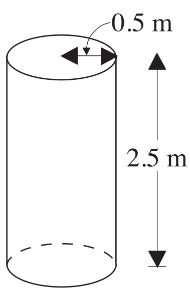


(d)



(e)



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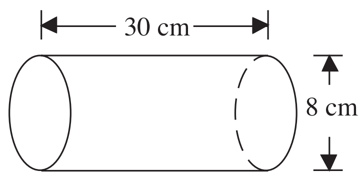
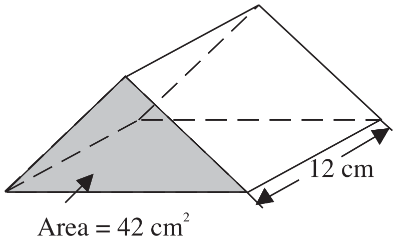
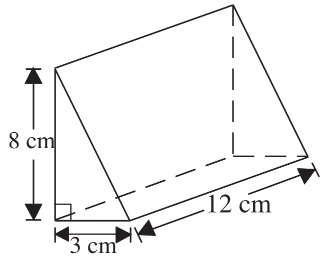


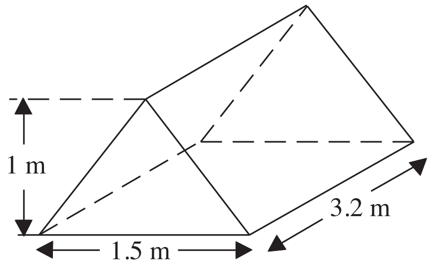
(g)



(h)

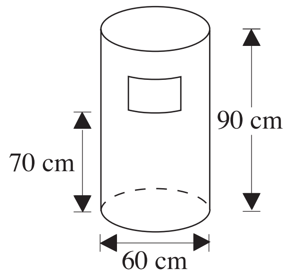


(i)



2. (a) Find the volume of the litter bin shown in the diagram, in m^3 to 2 decimal places.

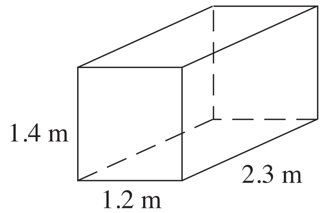
   (b) Find the volume of rubbish that can be put in the bin, if it must all be below the level of the hole in the side, in m^3 to 2 decimal places.



3. A water tank has the dimensions shown in the diagram.

   (a) Find the volume of the tank.

   (b) If the depth of water is 1.2 m, find the volume of the water.



4. A concrete pillar is a cylinder with a radius of 20 cm and a height of 2 m.

   (a) Find the volume of the pillar.

   The pillar is made of concrete, but contains 10 steel rods of length 1.8 m and diameter 1.2 cm.

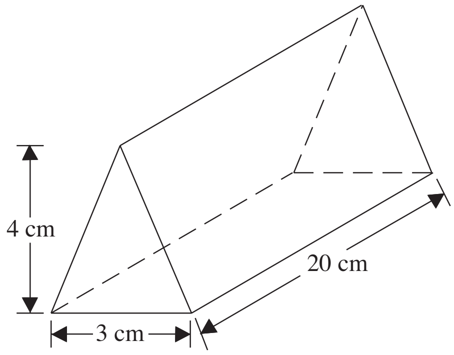
   (b) Find the volume of one of the rods and the volume of steel in the pillar.

   (c) Find the volume of concrete contained in the pillar.

5. The box shown in the diagram contains chocolate.

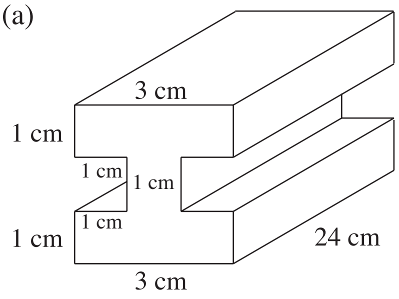
   (a) Find the volume of the box.

   (b) If the box contains 15 cm^3 of air, find the volume of the chocolate.

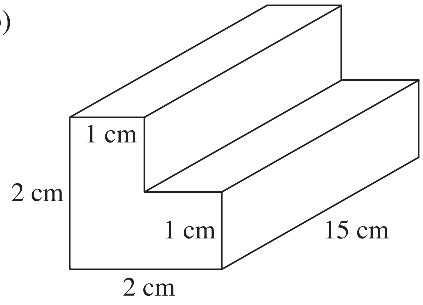


6. Find the volume of each prism below.

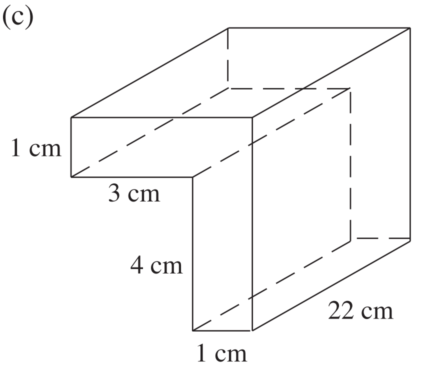
(a)



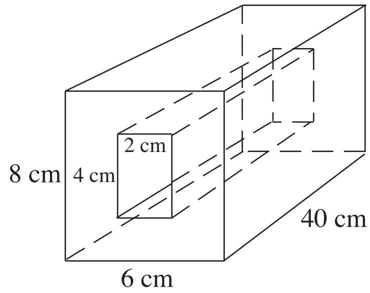
(b)



(c)

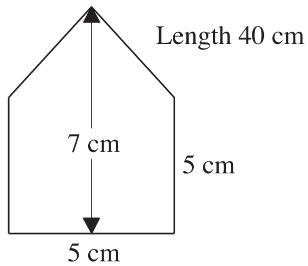


(d)

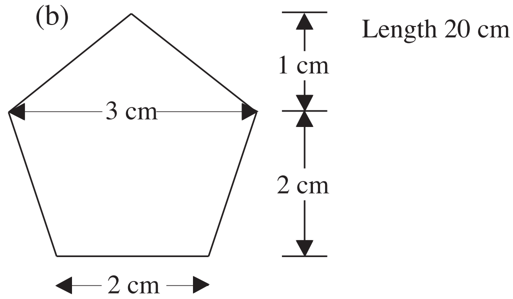


7. Each diagram below shows the cross section of a prism. Find the volume of the prism, given the length specified.

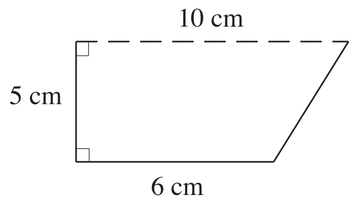
(a)



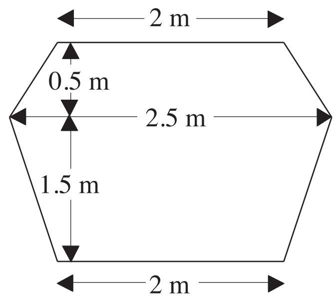
(b)



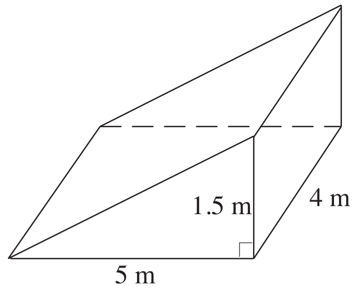
8. The diagram shows the cross section of a length of guttering. Find the maximum volume of water that a 5 m length of guttering could hold.



9. The diagram shows the cross section of a skip that is 15 m in length and is used to deliver sand to building sites. Find the volume of sand in the skip when it is filled level to the top.



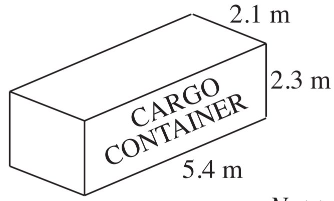
10. A ramp is constructed from concrete. Find the volume of concrete contained in the ramp.



Not to scale

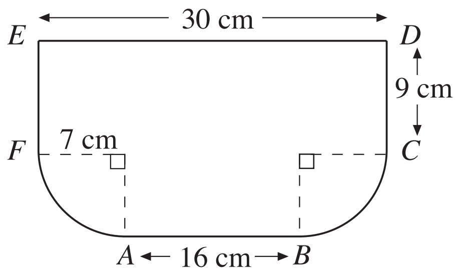
11. The diagram shows a cargo container.

Calculate the volume of the container.



Not to scale

12.



The diagram above, **not drawn to scale**, shows *ABCDEF,* a vertical cross-section of a container with *ED* being the top edge. *DC* and *EF* are vertical edges.

BC and *AF* are arcs of a circle of radius 7 cm and $AB \,||\, ED$.

$ED = 30$ cm; $AB = 16$ cm; $EF = DC = 9$ cm.

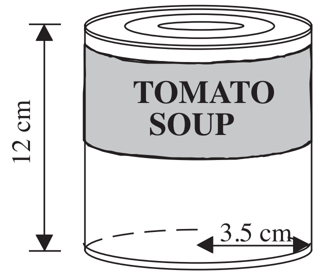
(i) Taking $\pi = \dfrac{22}{7}$, show that the area of *ABCDEF* is 459 cm^2.

(ii) Water is poured into the container until the water level is 4 cm from the top. If the container is 40 cm long and has uniform cross-section, calculate, to the nearest litre, the volume of water in the container.

(CXC)

13. Tomato soup is sold in cylindrical tins.

Each tin has a base radius of 3.5 cm and a height of 12 cm.

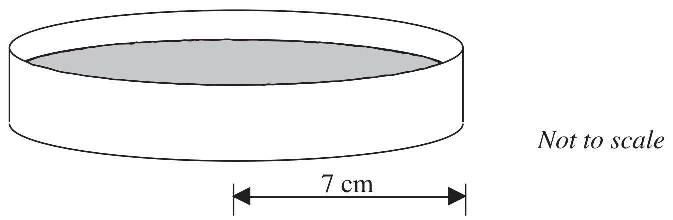


Not to scale

(a) Calculate the volume of soup in a full tin.

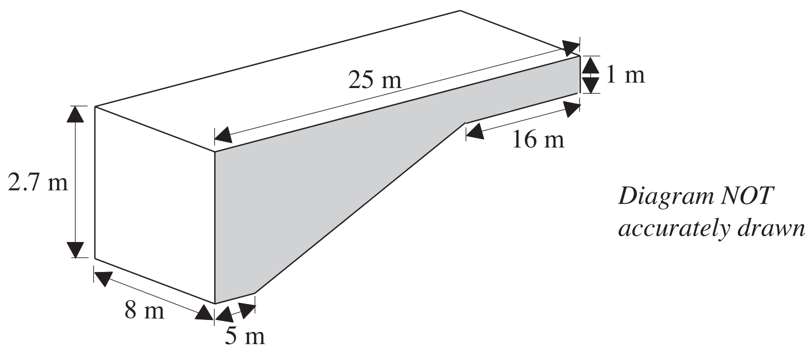
Take π to be 3.14 or use the π key on your calculator.

(b) Bradley has a full tin of tomato soup for dinner. He pours the soup into a cylindrical bowl of radius 7 cm.



What is the depth of the soup in the bowl?

14.



The diagram represents a swimming pool.
The pool has vertical sides.
The pool is 8 m wide.

(a) Calculate the area of the shaded cross section.

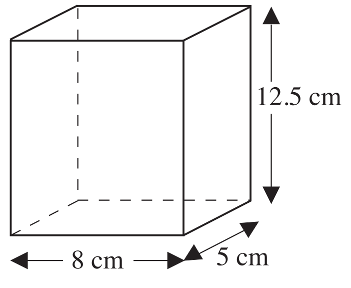
The swimming pool is completely filled with water.

(b) Calculate the volume of water in the pool.

64 m^3 leaks out of the pool.

(c) Calculate the distance by which the water level falls.

15. The diagram represents a carton in the shape of a cuboid.



(a) Calculate the volume of the carton.

There are 125 grams of sweets in a full carton.

John has to design a new carton that will contain 100 grams of sweets when it is full.

(b) (i) Work out the volume of the new carton.

(ii) Express the weight of the new carton as a percentage of the weight of the carton shown.

The new carton is in the shape of a cuboid.

The base of the new carton measures 7 cm by 6 cm.

(c) (i) Work out the area of the base of the new carton.

(ii) Calculate the height of the new carton.

14.2 Mass, Volume and Density

Density is a term used to describe the mass of a unit of volume of a substance. For example, if the density of a metal is 2000 kg/m^3, then 1 m^3 of the substance has a mass of 2000 kg.

Mass, volume and *density* are related by the following equations.

$$\text{Mass} = \text{Volume} \times \text{Density}$$

$$\text{Volume} = \frac{\text{Mass}}{\text{Density}}$$

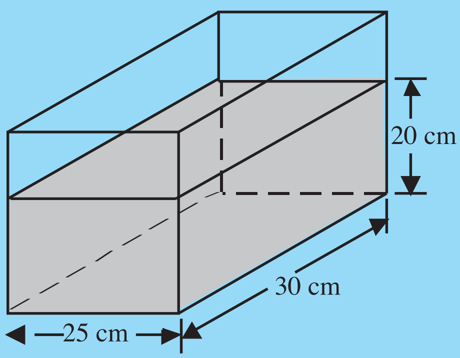
or

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}$$

The density of water is 1 gram / cm^3 (or 1 g cm^{-3}) or 1000 kg / m^3 (or 1000 kg m^{-3}).

Worked Example 1

Find the mass of water in the fish tank shown in the diagram.



Solution

First calculate the volume of water.

$$V = 25 \times 30 \times 20$$
$$= 15000 \text{ cm}^3$$

Now use

$$\text{Mass} = \text{Volume} \times \text{Density}$$
$$= 15000 \times 1 \quad (\text{as density of water} = 1\text{g} / \text{cm}^3)$$
$$= 15000 \text{ grams}$$
$$= 15 \text{ kg}$$

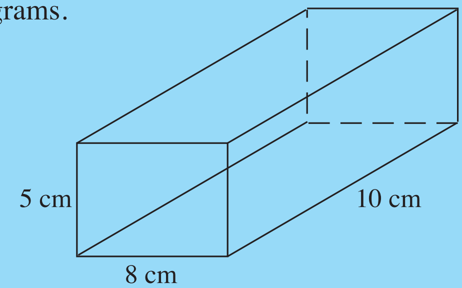
Worked Example 2

The block of metal shown has a mass of 500 grams.

Find its density in

(a) g / cm^3,

(b) kg / m^3.



Solution

(a) First find the volume.

$$\text{Volume} = 5 \times 8 \times 10$$
$$= 400 \text{ cm}^3$$

Then use

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}$$

$$\text{Density} = \frac{500}{400}$$
$$= 1.25 \text{ g/cm}^3$$

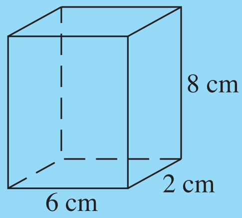
(b) The process can then be repeated working in kg and m.

$$\text{Volume} = 0.05 \times 0.08 \times 0.1$$
$$= 0.0004 \text{ m}^3$$

$$\text{Density} = \frac{0.5}{0.0004}$$
$$= \frac{5000}{4}$$
$$= 1250 \text{ kg} / \text{m}^3$$

Exercises

1. A drinks carton is a cuboid with size as shown.

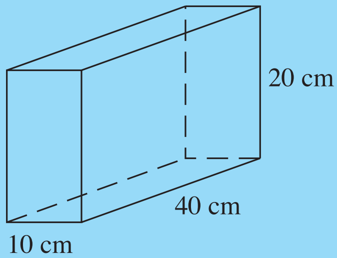


   (a) Find the volume of the carton.

   (b) If it contains 8 cm^3 of air, find the volume of the drink.

   (c) Find the mass of the drink if it has a density of 1 gram / cm^3.

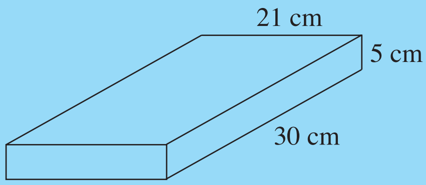
2. The diagram shows a concrete block of mass 6 kg.



   (a) Find the volume of the block.

   (b) What is the density of the concrete?

3. A ream (500 sheets) of paper is shown in the diagram.



   If the mass of the ream is 2.5 kg, find the density of the paper.

4. A barrel is a cylinder with radius 40 cm and height 80 cm. It is full of water.

   (a) Find the volume of the barrel.

   (b) Find the mass of the water in the barrel.

5. A metal bar has a cross section with an area of 3 cm^2 and a length of 40 cm. Its mass is 300 grams.

   (a) Find the volume of the bar.

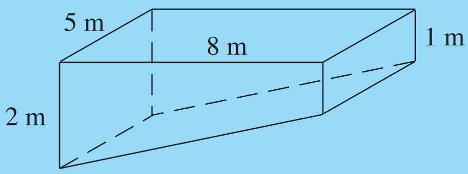
   (b) Find the density of the bar.

   (c) Find the mass of another bar with the same cross section and length 50 cm.

   (d) Find the mass of a bar made from the same material, but with a cross section of area 5 cm^2 and length 80 cm.

6. A bottle which holds 450 cm^3 of water has a mass of 530 grams. What is the mass of the empty bottle?

7. The diagram shows the dimensions of a swimming pool.



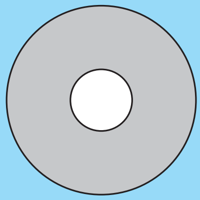
   (a) Find the volume of the swimming pool.

   (b) Find the mass of water in the pool if it is completely full.

   (c) In practice, the level of the water is 20 cm below the top of the pool.

   Find the volume and mass of the water in this case.

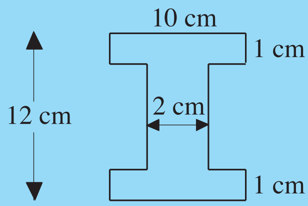
8. The density of a metal is 3 grams / cm^3. It is used to make a pipe with external radius of 1.5 cm and an internal radius of 0.5 cm.



   (a) Find the area of the cross section of the pipe.
   (b) If the length of the pipe is 75 cm, find its mass.
   (c) Find the length, to the nearest cm, of a pipe that has a mass of 750 grams.

9. A foam ball has a mass of 200 grams and a radius of 10 cm.
   (a) Use the formula $V = \frac{4\pi r^3}{3}$ to find the volume of the ball.
   (b) Find the density of the ball.
   (c) The same type of foam is used to make a cube with sides of length 12 cm. Find the mass of the cube.

10. The diagram shows the cross section of a metal beam. A 2 m length of the beam has a mass of 48 kg.



   (a) Find the density of the metal.

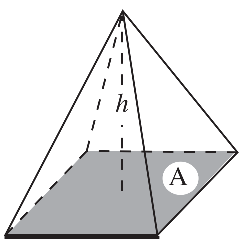
   A second type of beam uses the same type of metal, but all the dimensions of the cross section are increased by 50%.

   (b) Find the length of a beam of this type that has the same mass as the first beam.

14.3 Volumes of Pyramids, Cones and Spheres

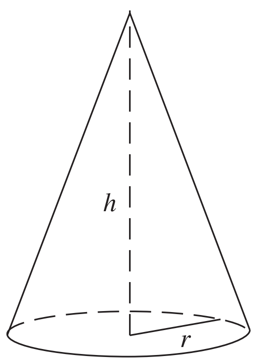
The volumes of a pyramid, a cone and a sphere are found using the following formulae.

Pyramid



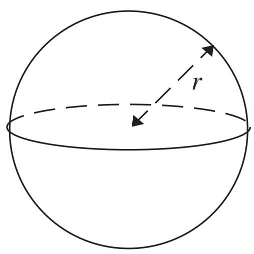
$V = \frac{1}{3}Ah$

Cone



$V = \frac{1}{3}\pi r^2 h$

Sphere



$V = \frac{4}{3}\pi r^3$

The proofs of these results are rather more complex and require mathematical analysis beyond the scope of this text.

Worked Example 1

A cone and sphere have the same radius of 12 cm. Find the height of the cone if the cone and sphere have the same volume.

Solution

Suppose that the height of the cone is h cm.

$$\text{Volume of cone} = \frac{1}{3}\pi \times 12^2 \times h = 48\pi h \ \text{cm}^3$$

$$\text{Volume of sphere} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(12)^3 = 2304\pi \ \text{cm}^3$$

Since the volumes are equal

$$48\pi h = 2304\pi$$

Solving for h,

$$h = \frac{2304\pi}{48\pi} = \frac{2304}{48} = 48 \text{ cm}$$

Exercises

1. Find the volume of the following containers:

   (a) a cylinder of base radius 3 cm and height 10 cm,

   (b) a cone of base radius 5 cm and height 12 cm,

   (c) a cone of base radius 7 cm and height 5 cm,

   (d) a cone of base radius 1 m and height 1.5 m,

   (e) a cone of base radius 9 cm and slant height 15 cm,

   (f) a cone of base diameter 20 cm and slant height 25 cm,

   (g) a sphere of radius 6 cm,

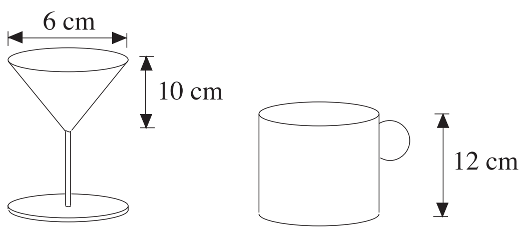
   (h) a hemisphere of radius 5 cm,

   (i) a pyramid of square base 10 cm and height 15 cm,

   (j) a pyramid of rectangular base 8 cm by 6 cm and height 9 cm.

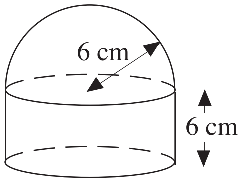
2. Calculate the radius of a sphere which has the same volume as a solid cylinder of base radius 5 cm and height 12 cm.

3. A wine glass is in the shape of a cone on a stem. The cylindrical tumbler is used to fill the wine glass. Find the diameter of the tumbler so that it has the same volume as the wine glass.

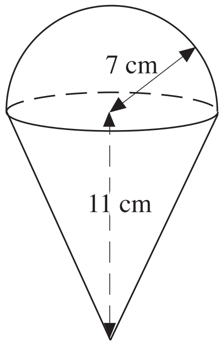


4. Find the volume of the following shapes made up of cones, hemispheres and cylinders.

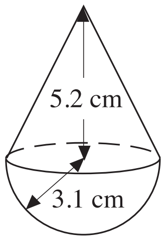
(a)



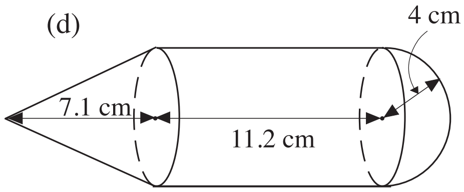
(b)



(c)

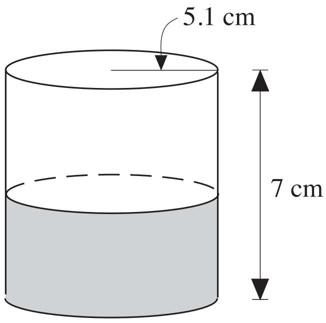


(d)

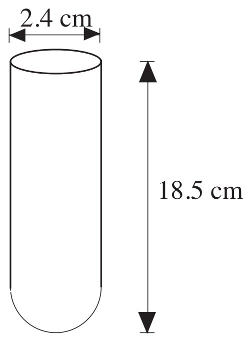


5. A cylinder is half filled with water as shown. A heavy sphere of diameter 2.5 cm is placed in the cylinder and sinks to the bottom.

By how much does the water rise in the cylinder?



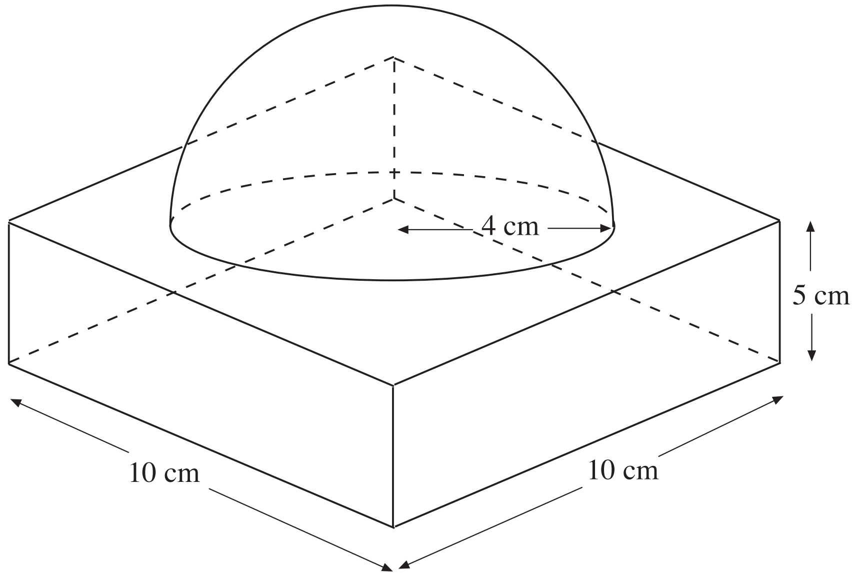
6. A test tube is in the shape of a hollow cylinder and hollow hemisphere. Calculate the volume of a liquid that can be held in the test tube.



7. The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

Calculate the volume of a sphere of radius 1.7 cm, giving your answer correct to 1 decimal place.

8. A marble paperweight consists of a cuboid and a hemisphere as shown in the diagram. The hemisphere has a radius of 4 cm.



Calculate the volume of the paperweight.

Challenge

A man wishes to take 4 litres of water out of a large tank of water, but he has only one 5-litre and one 3-litre jar. How can he do it?

Information

Archimedes (287BC-212BC), a Greek Mathematician, was once entrusted with the task of finding out whether the King's crown was made of pure gold. While taking his bath, he came up with a solution and was so excited that he dashed out into the street naked shouting "Eureka" (I have found it). The container that you use in the Science laboratory to measure the volume of an irregular object is known as an Eureka can (named after this incident). Archimedes was so engrossed in his work that when his country was conquered by the Romans, he was still working hard at his mathematics. When a Roman soldier ordered him to leave his desk, Archimedes replied, "Don't disturb my circles." He was killed by that soldier for disobeying orders.

Archimedes' greatest contribution was the discovery that the volume of a sphere is $\frac{2}{3}$ *that of a cylinder whose diameter is the same as the diameter of the sphere. At his request, the sphere in the cylinder diagram was engraved on his tombstone.*