# STRAND B: Number Theory

# Unit 7 *Number System and Bases*

## Student Text

Contents

**Section**

# 7 Number System and Bases

## 7.1 Number System

The number system is classified into various categories.

***Natural numbers (or 'counting numbers')***
These are the positive whole numbers, namely 1, 2, 3,,, ...

***Integers***
These are the set of positive and negative whole numbers, e.g. 1, 10, 364, – 7, – 102.
The set of integers is denoted by Z.

***Rational Numbers***
A *rational* number is a number which can be written in the form

$$\frac{m}{n}, \quad \text{where } m \text{ and } n \text{ are integers.}$$

For example, $\frac{4}{5}$ is a rational number. A rational number is in its simplest form if $m$ and $n$ have no common factor and $n$ is positive. The set of rational numbers is denoted by Q.

***Irrational Numbers***
There are numbers which *cannot* be written in the form

$$\frac{m}{n}, \quad \text{where } m \text{ and } n \text{ are integers.}$$

Examples of irrational numbers are $\pi$, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$.

***Terminating Decimals***
These are decimal numbers which stop after a certain number of decimal places.
For example,

$$\frac{7}{8} = 0.875$$

is a terminating decimal because it stops (terminates) after 3 decimal places.

***Recurring Decimals***
These are decimal numbers which keep repeating a digit or group of digits; for example

$$\frac{137}{259} = 0.528\,957\,528\,957\,528\,957\,....$$

is a recurring decimal. The six digits 528957 repeat in this order. Recurring decimals are written with dots over the first and last digit of the repeating digits, e.g. $0.\dot{5}28\,95\dot{7}$

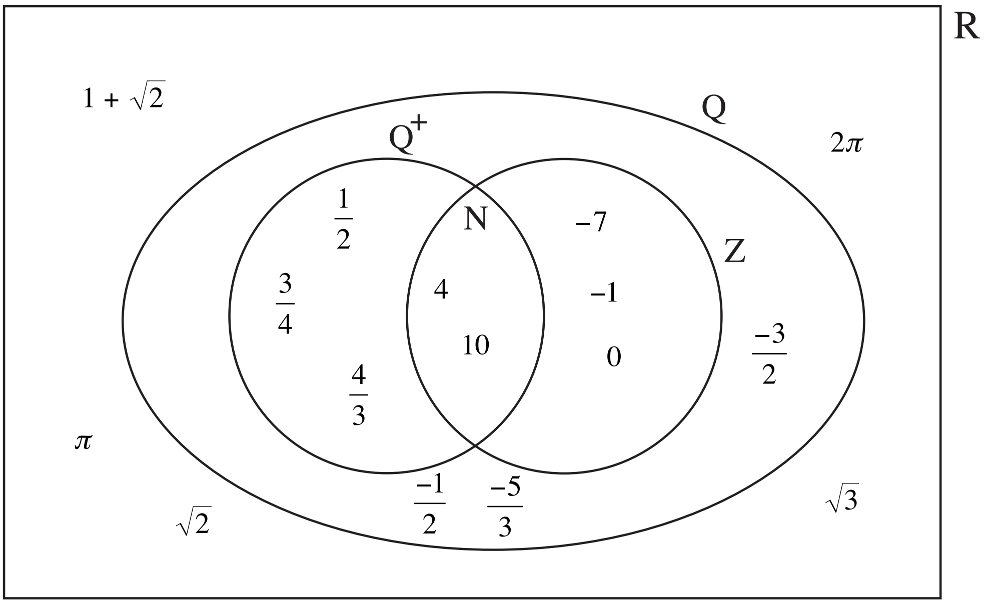
## Note

All *terminating* and *recurring* decimals can be written in the form $\frac{m}{n}$, so they are *rational* numbers.

***Real Numbers***

These are made up of all possible *rational* and *irrational* numbers; the set of real numbers is denoted by R.

We can use a Venn diagram to illustrate this classification of number, as shown here.




R : Real numbers

N : Natural numbers

Z : Integers

Q : Rational numbers

$Q^+$ : Positive rational numbers

Some numbers have been inserted to illustrate each set.

## Worked Example 1

Classify the following numbers as integers, rational, irrational, recurring decimals, terminating decimals.

$$\frac{5}{7},\quad -7,\quad 0.6,\quad 0.4\dot{1}21\dot{3},\quad \frac{5}{8},\quad 11,\quad \sqrt{10},\quad \frac{\pi}{4},\quad \sqrt{49}$$

## Solution

| Number | Rational | Irrational | Integer | Recurring Decimal | Terminating Decimal |
|---|---|---|---|---|---|
| $\frac{5}{7}$ | ✓ | | | $0.\dot{7}1428\dot{5}$ | |
| $-7$ | ✓ | | ✓ | | |
| 0.6 | ✓ | | | | ✓ |
| $0.4\dot{1}21\dot{3}$ | ✓ | | | ✓ | |
| $\frac{5}{8}$ | ✓ | | | | 0.625 |
| 11 | ✓ | | ✓ | | |
| $\sqrt{10}$ | | ✓ | | | |
| $\frac{\pi}{4}$ | | ✓ | | | |
| $\sqrt{49}$ | ✓ | | ✓ | | |

## Worked Example 2

Show that the numbers 0.345 and $0.0\dot{9}1\dot{7}$ are rational.

## Solution

For a terminating decimal the proof is straightforward.

$$0.345 = \frac{345}{1000} = \frac{69}{200}$$

which is rational because $m = 69$ and $n = 200$ are integers.

For a recurring decimal, we multiply by a power of ten so that after the decimal point we have only the repeating digits.

$$0.0\dot{9}1\dot{7} \times 10\,000 = 917.917\,917\,917\,...$$

Also

$$0.0\dot{9}1\dot{7} \times 10 = 0.917\,917\,917\,...$$

Subtracting the second equation from the first gives

$$0.0\dot{9}1\dot{7} \times 10\,000 - 0.0\dot{9}1\dot{7} \times 10 = 917$$

$$0.0\dot{9}1\dot{7} \times (10000 - 10) = 917$$

$$0.0\dot{9}1\dot{7} = \frac{917}{9990}$$

which is rational because $m = 917$ and $n = 9990$ .

## Worked Example 3

Prove that $\sqrt{3}$ is irrational.

## **Solution**

Assume that $\sqrt{3}$ is rational so we can find integers $m$ and $n$ such that $\sqrt{3} = \frac{m}{n}$, and $m$ and $n$ have no common factors. Square both sides,

$$3 = \frac{m^2}{n^2}$$

Multiply both sides by $n^2$,

$$m^2 = 3n^2$$

Since $3n^2$ is divisible by 3, then $m^2$ is divisible by 3.

Since $m$ is an integer, $m^2$ is an integer and 3 is prime, then $m$ must be divisible by 3.

Let $m = 3p$, where $p$ is an integer.

$$3n^2 = m^2 = (3p)^2 = 9p^2$$

$$\Rightarrow \quad n^2 = 3p^2$$

Since $3p^2$ is divisible by 3, $n^2$ is divisible by 3 and hence $n$ is divisible by 3.

We have shown that $m$ and $n$ are both divisible by 3.

This contradicts the original assumption that $\sqrt{3} = \frac{m}{n}$, where $m$ and $n$ are integers with no factor common. Our original assumption is wrong. $\sqrt{3}$ is *not* rational – it is irrational.

## Exercises

1. Classify the following numbers as rational or irrational, terminating or recurring decimals.

| | | |
|---|---|---|
| (a) $\sqrt{100}$ | (b) $0.\dot{6}$ | (c) $\frac{\pi}{0.2}$ |
| (d) $\frac{13}{99}$ | (e) 0.75 | (f) $\frac{1}{\pi}$ |
| (g) $\sqrt{11}$ | (h) $\frac{1.6}{4}$ | (i) $\pi^2$ |
| (j) $\frac{5}{11}$ | (k) $\sin 45°$ | (l) $\cos 60°$ |

2. Where possible, write each of the following numbers in the form $\dfrac{m}{n}$, where $m$ and $n$ are integers with no common factors

(a) $0.49$ (b) $0.\dot{3}$ (c) $\dfrac{\sqrt{49}}{4}$ (d) $\sqrt{7}$

(e) $0.417$ (f) $0.\dot{1}$ (g) $0.\dot{0}\dot{9}$ (h) $\dfrac{\sqrt{36}}{\sqrt{121}}$

(i) $0.125$ (j) $0.962$

3. Write each of the following recurring decimals in the form $\dfrac{m}{n}$, where $m$ and $n$ are integers with no common factors.

(a) $0.\dot{4}\dot{1}$ (b) $0.0\dot{4}0\dot{2}$ (c) $0.\dot{1}4285\dot{7}$ (d) $0.\dot{8}$

(e) $0.\dot{8}1\dot{2}$ (f) $0.\dot{5}$ (g) $0.\dot{9}0\dot{9}$

4. Given the rational numbers

$$a = \frac{2}{3}, \quad b = \frac{7}{9}, \quad c = \frac{11}{15}, \quad d = \frac{1}{8}$$

show that $a + b$, $c + d$, $a + c + d$, $ab$, $cd$, $bc$ and $abc$ are rational numbers.

5. Prove that the following numbers are irrational,

$$\sqrt{2}, \quad \sqrt{5}, \quad \sqrt{11}$$

6. What happens if you try to prove that $\sqrt{4}$ is irrational using the proof by a contradiction method.

7. Write down two rational numbers and two irrational numbers which lie in each of the following intervals.

(a) $0 < x < 1$ (b) $1 < x < 9$ (c) $-4 < x < -2$ (d) $98 < x < 99$

8. Which of these expressions can be written as recurring decimals and which can be written as non-recurring decimals?

$$\frac{2}{7}, \quad \pi, \quad \frac{1}{17}, \quad \sqrt{7}, \quad \frac{6}{47}$$

9. $$x = \sqrt{a^2 + b^2}$$

State whether $x$ is rational or irrational in each of the following cases, and show sufficient working to justify each answer.

(a) $a = 5$ and $b = 12$ (b) $a = 5$ and $b = 6$

(c) $a = \sqrt{2}$ and $b = \sqrt{7}$ (d) $a = \dfrac{3}{7}$ and $b = \dfrac{4}{7}$

10. (a) Write down *two* irrational numbers that multiply together to give a rational number.

(b) Which of the following numbers are rational?

$$2^{\frac{1}{2}},\quad 2^{-2},\quad 4^{\frac{1}{2}},\quad 4^{-2},\quad \pi^{\frac{1}{2}},\quad \pi^{-2}$$

11. (a) A Mathematics student attempted to define an *irrational number* as follows:

"An irrational number is a number which, in its decimal form, goes on and on."

(i) Give an example to show that this definition is not correct.

(ii) What must be added to the definition to make it correct?

(b) Which of the following are rational and which are irrational?

$$\sqrt{4\frac{1}{4}},\quad \sqrt{6\frac{1}{4}},\quad \frac{1}{3}+\sqrt{3},\quad \left(\frac{1}{3}\sqrt{3}\right)^2$$

Express each of the rational numbers in the form $\frac{p}{q}$, where $p$ and $q$ are integers.

12. (a) Write down an irrational number that lies between 4 and 5.

(b) $N$ is a rational number which is not equal to zero.

Show clearly why $\frac{1}{N}$ must also be rational.

13. $n$ is a positive integer such that $\sqrt{n} = 15.4$ correct to 1 decimal place.

(a) (i) Find a value of $n$.

(ii) Explain why $\sqrt{n}$ is irrational.

(b) Write down a number between 10 and 11 that has a rational square root.

# 7.2 Binary Numbers

We normally work with numbers in base 10. In this section we consider numbers in *base 2*, often called *binary numbers*.

In *base 10* we use the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

In *base 2* we use only the digits 0 and 1.

Binary numbers are at the heart of all computing systems since, in an electrical circuit, 0 represents *no* current flowing whereas 1 represents a current flowing.

In *base 10* we use a system of place values as shown below:

| 1000 | 100 | 10 | 1 | | |
|---|---|---|---|---|---|
| 4 | 2 | 1 | 5 | → | $4 \times 1000 + 2 \times 100 + 1 \times 10 + 5 \times 1$ |
| 3 | 1 | 0 | 2 | → | $3 \times 1000 + 1 \times 100 + 2 \times 1$ |

Note that, to obtain the place value for the next digit to the left, we multiply by 10. If we were to add another digit to the front (left) of the numbers above, that number would represent 10 000s.

In *base 2* we use a system of place values as shown below:

| 64 | 32 | 16 | 8 | 4 | 2 | 1 | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | → | $1 \times 64 = 64$ |
| 1 | 0 | 0 | 1 | 0 | 0 | 1 | → | $1 \times 64 + 1 \times 8 + 1 \times 1 = 73$ |

Note that the place values begin with 1 and are multiplied by 2 as you move to the left.

Once you know how the place value system works, you can convert binary numbers to base 10, and vice versa.

## Worked Example 1

Convert the following binary numbers to base 10:

(a) 111 (b) 101 (c) 1100110

## Solution

For each number, consider the place value of every digit.

(a)

| 4 | 2 | 1 | | |
|---|---|---|---|---|
| 1 | 1 | 1 | → | $4 + 2 + 1 = 7$ |

The binary number 111 is 7 in base 10.

(b)

| 4 | 2 | 1 | | |
|---|---|---|---|---|
| 1 | 0 | 1 | → | $4 + 1 = 5$ |

The binary number 101 is 5 in base 10.

(c)

| 64 | 32 | 16 | 8 | 4 | 2 | 1 | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 1 | 1 | 0 | → | $64 + 32 + 4 + 2 = 102$ |

The binary number 1100110 is 102 in base 10.

## Worked Example 2

Convert the following base 10 numbers into binary numbers:

(a) 3 (b) 11 (c) 140

## **Solution**

We need to write these numbers in terms of the binary place value headings 1, 2, 4, 8, 16, 32, 64, 128, ..., etc.

(a)

$$3 = 2 + 1 \rightarrow$$

| 2 | 1 |
|---|---|
| 1 | 1 |

The base 10 number 3 is written as 11 in base 2.

(b)

$$11 = 8 + 2 + 1 \rightarrow$$

| 8 | 4 | 2 | 1 |
|---|---|---|---|
| 1 | 0 | 1 | 1 |

The base 10 number 11 is written as 1011 in base 2.

(c)

$$140 = 128 + 8 + 4 \rightarrow$$

| 128 | 64 | 32 | 16 | 8 | 4 | 2 | 1 |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |

The base 10 number 140 is written as 10001100 in base 2.

## Exercises

1. Convert the following binary numbers to base 10:

(a) 110 (b) 1111 (c) 1001 (d) 1101
(e) 10001 (f) 11011 (g) 1111111 (h) 1110001
(i) 10101010 (j) 11001101 (k) 111000111 (l) 1100110

2. Convert the following base 10 numbers to binary numbers:

(a) 9 (b) 8 (c) 14 (d) 17
(e) 18 (f) 30 (g) 47 (h) 52
(i) 67 (j) 84 (k) 200 (l) 500

3. Convert the following base 10 numbers to binary numbers:

(a) 5 (b) 9 (c) 17 (d) 33

Describe any pattern that you notice in these binary numbers.
What will be the next base 10 number that will fit this pattern?

4. Convert the following base 10 numbers to binary numbers:

(a) 3 (b) 7 (c) 15 (d) 31

What is the next base 10 number that will continue your binary pattern?

5. A particular binary number has 3 digits.
   (a) What are the *largest* and *smallest* possible binary numbers?
   (b) Convert these numbers to base 10.

6. When a particular base 10 number is converted it gives a 4-digit binary number. What could the original base 10 number be?

7. A 4-digit binary number has 2 zeros and 2 ones.
   (a) List all the possible binary numbers with these digits.
   (b) Convert these numbers to base 10.

8. A binary number has 8 digits and is to be converted to base 10.
   (a) What is the *largest* possible base 10 answer?
   (b) What is the *smallest* possible base 10 answer?

9. The base 10 number 999 is to be converted to binary. How many more digits does the binary number have than the number in base 10?

10. Calculate the difference between the *base 10* number 11111 and the *binary* number 11111, giving your answer in base 10.

# 7.3 Adding and Subtracting Binary Numbers

It is possible to add and subtract binary numbers in a similar way to base 10 numbers.
For example, $1 + 1 + 1 = 3$ in base 10 becomes $1 + 1 + 1 = 11$ in binary.
In the same way, $3 - 1 = 2$ in base 10 becomes $11 - 1 = 10$ in binary. When you add and subtract binary numbers you will need to be careful when 'carrying' or 'borrowing' as these will take place more often.

Key *Addition* Results for Binary Numbers

$1 + 0 = 1$

$1 + 1 = 10$

$1 + 1 + 1 = 11$

Key *Subtraction* Results for Binary Numbers

$1 - 0 = 1$

$10 - 1 = 1$

$11 - 1 = 10$

## Worked Example 1

Calculate, using binary numbers:

(a) 111 + 100 (b) 101 + 110 (c) 1111 + 111

## Solution

(a)
```
   1 1 1
 + 1 0 0
 -------
 1 0 1 1
 -------
 1
```

(b)
```
   1 0 1
 + 1 1 0
 -------
 1 0 1 1
 -------
 1
```

(c)
```
   1 1 1 1
 +   1 1 1
 ---------
 1 0 1 1 0
 ---------
   1 1 1
```

Note how important it is to 'carry' correctly.

## Worked Example 2

Calculate the binary numbers:

(a) 111 – 101 (b) 110 – 11 (c) 1100 – 101

## Solution

(a)
```
   1 1 1
 – 1 0 1
 -------
     1 0
 -------
```

(b)
```
   1 1 0
 –   1 1
 -------
     1 1
 -------
```

(c)
```
   1 1 0 0
 –   1 0 1
 ---------
     1 1 1
 ---------
```

## Exercises

1. Calculate the binary numbers:

| | | |
|---|---|---|
| (a) 11 + 1 | (b) 11 + 11 | (c) 111 + 11 |
| (d) 111 + 10 | (e) 1110 + 111 | (f) 1100 + 110 |
| (g) 1111 + 10101 | (h) 1100 + 11001 | (i) 1011 + 1101 |
| (j) 1110 + 10111 | (k) 1110 + 1111 | (l) 11111 + 11101 |

2. Calculate the binary numbers:

| | | |
|---|---|---|
| (a) 11 – 10 | (b) 110 – 10 | (c) 1111 – 110 |
| (d) 100 – 10 | (e) 100 – 11 | (f) 1000 – 11 |
| (g) 1101 – 110 | (h) 11011 – 110 | (i) 1111 – 111 |
| (j) 110101 – 1010 | (k) 11011 – 111 | (l) 11110 – 111 |

3. Calculate the binary numbers:

(a) $11 + 11$ (b) $111 + 111$

(c) $1111 + 1111$ (d) $11111 + 11111$

Describe any patterns that you observe in your answers.

4. Calculate the binary numbers:

(a) $10 + 10$ (b) $100 + 100$

(c) $1000 + 1000$ (d) $10000 + 10000$

Describe any patterns that you observe in your answers.

5. Solve the following equations, where all numbers, including $x$, are binary:

(a) $x + 11 = 1101$ (b) $x - 10 = 101$

(c) $x - 1101 = 11011$ (d) $x + 1110 = 10001$

(e) $x + 111 = 11110$ (f) $x - 1001 = 11101$

6. Calculate the binary numbers:

(a) $10 - 1$ (b) $100 - 1$

(c) $1000 - 1$ (d) $10000 - 1$

Describe any patterns that you observe in your answers.

7. (a) Convert the binary numbers 11101 and 1110 to base 10.

(b) Add together the two base 10 numbers.

(c) Add together the two binary numbers.

(d) Convert your answer to base 10 and compare with your answer to (b).

8. (a) Convert the binary numbers 11101 and 10111 to base 10.

(b) Calculate the difference between the two base 10 numbers.

(c) Convert your answer to (b) into a binary number.

(d) Calculate the difference between the two binary numbers and compare with your answer to (c).

9. Here are 3 binary numbers:

1110101 1011110 1010011

Working in binary,

(a) add together the *two smaller* numbers,

(b) add together the *two larger* numbers,

(c) take the smallest number away from the largest number,

(d) add together all three numbers.

10. Calculate the binary numbers:

    (a) $111 + 101 + 100$

    (b) $11101 + 10011 + 110111$

# 7.4 Multiplying Binary Numbers

Long multiplication can be carried out with binary numbers and is explored in this section. Note that multiplying by numbers like 10, 100 and 1000 is very similar to working with base 10 numbers.

## Worked Example 1

Calculate the binary numbers:

(a) $1011 \times 100$ (b) $110110 \times 1000$ (c) $11011 \times 10000$

Check your answers to (a) and (c) by converting each number to base 10.

## Solution

(a) $1011 \times 100 = 101100$ (b) $110110 \times 1000 = 110110000$

(c) $11011 \times 10000 = 110110000$

Checking:

(a)

| 8 | 4 | 2 | 1 | | |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | $\rightarrow$ | $8 + 2 + 1 = 11$ |

| 4 | 2 | 1 | | |
|---|---|---|---|---|
| 1 | 0 | 0 | $\rightarrow$ | 4 |

| 32 | 16 | 8 | 4 | 2 | 1 | | |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 | 0 | $\rightarrow$ | $32 + 8 + 4 = 44$ |

and $11 \times 4 = 44$, as expected.

(c)

| 16 | 8 | 4 | 2 | 1 | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 1 | $\rightarrow$ | $16 + 8 + 2 + 1 = 27$ |

| 16 | 8 | 4 | 2 | 1 | | |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | $\rightarrow$ | 16 |

| 256 | 128 | 64 | 32 | 16 | 8 | 4 | 2 | 1 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |

$\rightarrow 256 + 128 + 32 + 16 = 432$

and $27 \times 16 = 432$, as expected.

*Note*: clearly it is more efficient to keep the numbers in binary when doing the calculations.

## Worked Example 2

Calculate the binary numbers:

(a) $1011 \times 11$ (b) $1110 \times 101$

(c) $11011 \times 111$ (d) $11011 \times 1001$

## Solution

(a)
$$\begin{array}{r} 1011 \\ \times \quad 11 \\ \hline 1011 \\ 10110 \\ \hline 100001 \\ \hline {\scriptstyle 1\,1\,1\,1\quad} \end{array}$$

(b)
$$\begin{array}{r} 1110 \\ \times \quad 101 \\ \hline 1110 \\ 111000 \\ \hline 1000110 \\ \hline {\scriptstyle 1\,1\,1\qquad\quad} \end{array}$$

(c)
$$\begin{array}{r} 11011 \\ \times \quad 111 \\ \hline 11011 \\ 110110 \\ 1101100 \\ \hline 10111101 \\ \hline {\scriptstyle 1\,1\,1\,1\,1\,1\quad} \end{array}$$

(d)
$$\begin{array}{r} 11011 \\ \times \quad 1001 \\ \hline 11011 \\ 11011000 \\ \hline 11110011 \\ \hline {\scriptstyle 1\,1\qquad\quad} \end{array}$$

## Exercises

1. Calculate the binary numbers:

   (a) $111 \times 10$ (b) $1100 \times 100$

   (c) $101 \times 1000$ (d) $11101 \times 1000$

   (e) $11000 \times 10$ (f) $10100 \times 1000$

   (g) $10100 \div 10$ (h) $1100 \div 100$

   Check your answers by converting to base 10 numbers.

2. Calculate the binary numbers:

(a) $111 \times 11$ (b) $1101 \times 11$

(c) $1101 \times 101$ (d) $1111 \times 110$

(e) $11011 \times 1011$ (f) $11010 \times 1011$

(g) $10101 \times 101$ (h) $10101 \times 111$

(i) $10101 \times 110$ (j) $100111 \times 1101$

3. Solve the following equations, where all numbers, including $x$, are binary:

(a) $\frac{x}{11} = 110$ (b) $\frac{x}{101} = 101$

(c) $\frac{x}{10} = 111$ (d) $\frac{x}{111} = 1011$

4. Multiply each of the following binary numbers by itself:

(a) 11 (b) 111 (c) 1111

What do you notice about your answers to parts (a), (b) and (c)?

What will you get if you multiply 11111 by itself?

5. Multiply each of the following binary numbers by itself:

(a) 101 (b) 1001 (c) 10001 (d) 100001

What would you expect to get if you multiplied 1000001 by itself?

6. Calculate the binary numbers:

(a) $101\,(110 + 1101)$ (b) $1101\,(1111 - 110)$

(c) $111\,(1000 - 101)$ (d) $1011\,(10001 - 1010)$

7. Here are 3 binary numbers:

11011 11100 10011

Working in binary,

(a) multiply the two *larger* numbers,

(b) multiply the two *smaller* numbers.

8. (a) Multiply the base 10 numbers 45 and 33.

(b) Convert your answer to a binary number.

(c) Convert 45 and 33 to binary numbers.

(d) Multiply the binary numbers obtained in part (c) and compare this answer with your answer to part (b).

# 7.5 Other Bases

The ideas that we have considered can be extended to other number bases.

The table lists the digits used in some other number bases.

| *Base* | *Digits Used* |
|---|---|
| 2 | 0, 1 |
| 3 | 0, 1, 2 |
| 4 | 0, 1, 2, 3 |
| 5 | 0 1, 2, 3, 4 |

The powers of the base number give the place values when you convert to base 10. For example, for base 3, the place values are the powers of 3, i.e. 1, 3, 9, 27, 81, 243, etc. This is shown in the following example, which also shows how the base 3 number 12100 is equivalent to the base 10 number 144.

*Base 3*

| 81 | 27 | 9 | 3 | 1 |
|---|---|---|---|---|
| 1 | 2 | 1 | 0 | 0 |

$\rightarrow (1 \times 81) + (2 \times 27) + (1 \times 9) + (0 \times 3) + (0 \times 1) = 144$ in base 10

The following example shows a conversion from base 5 to base 10 using the powers of 5 as place values.

*Base 5*

| 625 | 125 | 25 | 5 | 1 |
|---|---|---|---|---|
| 4 | 1 | 0 | 0 | 1 |

$\rightarrow (4 \times 625) + (1 \times 125) + (0 \times 25) + (0 \times 5) + (1 \times 1) = 2626$ in base 10

## Worked Example 1

Convert each of the following numbers to base 10:

(a) 412 in base 6.

(b) 374 in base 9.

(c) 1432 in base 5.

## Solution

(a)

| 36 | 6 | 1 |
|---|---|---|
| 4 | 1 | 2 |

$\rightarrow (4 \times 36) + (1 \times 6) + (2 \times 1) = 152$ in base 10

(b)

| 81 | 9 | 1 |
|---|---|---|
| 3 | 7 | 4 |

$\rightarrow (3 \times 81) + (7 \times 9) + (4 \times 1) = 310$ in base 10

(c)

| 125 | 25 | 5 | 1 |
|---|---|---|---|
| 1 | 4 | 3 | 2 |

$\rightarrow (1 \times 125) + (4 \times 25) + (3 \times 5) + (2 \times 1) = 242$ in base 10

## Worked Example 2

Convert each of the following base 10 numbers to the base stated:

(a) 472 to base 4, (b) 179 to base 7, (c) 342 to base 3.

## Solution

(a) For base 4 the place values are 256, 64, 16, 4, 1, and you need to express the number 472 as a linear combination of 256, 64, 16, 4 and 1, but with no multiplier greater than 3.

We begin by writing

$472 = (1 \times 256) + 216$

The next stage is to write the remaining 216 as a linear combination of 64, 16, 4 and 1.

We use the fact that

$216 = (3 \times 64) + 24$

and, continuing in this way,

$24 = (1 \times 16) + 8$

$8 = (2 \times 4) + 0$

Putting all these stages together,

$$\begin{aligned} 472 &= (1 \times 256) + (3 \times 64) + (1 \times 16) + (2 \times 4) + (0 \times 1) \\ &= 13120 \text{ in base } 4 \end{aligned}$$

(b) For base 7 the place values are 49, 7, 1.

$$\begin{aligned} 179 &= (3 \times 49) + (4 \times 7) + (4 \times 1) \\ &= 344 \text{ in base } 7 \end{aligned}$$

(b) For base 3 the place values are 243, 81, 27, 9, 3, 1.

$$\begin{aligned} 342 &= (1 \times 243) + (1 \times 81) + (0 \times 27) + (2 \times 9) + (0 \times 3) + (0 \times 1) \\ &= 110200 \text{ in base } 3 \end{aligned}$$

## Worked Example 3

Carry out each of the following calculations in the base stated:

(a) $14 + 21$ base 5 (b) $16 + 32$ base 7

(c) $141 + 104$ base 5 (d) $212 + 121$ base 3

Check your answer in (a) by changing to base 10 numbers.

## Solution

(a)

$$\begin{array}{r} 1\ 4 \\ +\ 2\ 1 \\ \hline 4\ 0 \\ \hline {\scriptstyle 1\ \ } \end{array}$$

Note that $4 + 1 = 10$ in base 5.

(b)

$$\begin{array}{r} 1\ 6 \\ +\ 3\ 2 \\ \hline 5\ 1 \\ \hline {\scriptstyle 1\ \ } \end{array}$$

Note that $6 + 2 = 11$ in base 7.

(c)

$$\begin{array}{r} 1\ 4\ 1 \\ +\ 1\ 0\ 4 \\ \hline 3\ 0\ 0 \\ \hline {\scriptstyle 1\ 1\ \ } \end{array}$$

Note that $1 + 4 = 10$ in base 5.

(d)

$$\begin{array}{r} 2\ 1\ 2 \\ +\ 1\ 2\ 1 \\ \hline 1\ 1\ 1\ 0 \\ \hline {\scriptstyle 1\ 1\ 1\ \ } \end{array}$$

Note that, in base 3,

$$\begin{aligned} 2 + 1 &= 10 \\ 1 + 2 + 1 &= 11 \\ 2 + 1 + 1 &= 11 \end{aligned}$$

Checking in (a):

(a)

$$\begin{array}{cc} 5 & 1 \\ \hdashline 1 & 4 \end{array} \rightarrow (1 \times 5) + (4 \times 1) = 9$$

$$\begin{array}{cc} 5 & 1 \\ \hdashline 2 & 1 \end{array} \rightarrow (2 \times 5) + (1 \times 1) = 11$$

$$\begin{array}{cc} 5 & 1 \\ \hdashline 4 & 0 \end{array} \rightarrow (4 \times 5) + (0 \times 1) = 20$$

and $9 + 11 = 20$, as expected.

## Worked Example 4

Carry out each of the following multiplications in the base stated:

(a) $141 \times 23$ in base 5 (b) $122 \times 12$ in base 3

(c) $512 \times 24$ in base 6

Check your answer to (b) by converting to base 10 numbers.

## Solution

(a)

$$\begin{array}{r} 1\ 4\ 1 \\ \times \quad 2\ 3 \\ \hline 1\ 0\ 2\ 3 \\ 3\ 3\ 2\ 0 \\ \hline 4\ 3\ 4\ 3 \\ \hline \end{array}$$

Note that, in base 5,

$3 \times 4 = 22$

$2 \times 4 = 13$

(b)

$$\begin{array}{r} 1\ 2\ 2 \\ \times \quad 1\ 2 \\ \hline 1\ 0\ 2\ 1 \\ 1\ 2\ 2\ 0 \\ \hline 1\ 0\ 0\ 1\ 1 \\ \hline {\scriptstyle 1\ 1\ 1}\quad\quad \end{array}$$

Note that, in base 3,

$2 \times 2 = 11$

(c)

$$\begin{array}{r} 5\ 1\ 2 \\ \times \quad 2\ 4 \\ \hline 3\ 2\ 5\ 2 \\ 1\ 4\ 2\ 4\ 0 \\ \hline 2\ 1\ 5\ 3\ 2 \\ \hline {\scriptstyle 1\quad 1}\quad\quad \end{array}$$

Note that, in base 6,

$2 \times 4 = 12$

$4 \times 5 = 32$

$2 \times 5 = 14$

Checking in (b):

(b)

| 9 | 3 | 1 | |
|---|---|---|---|
| 1 | 2 | 2 | $\rightarrow (1 \times 9) + (2 \times 3) + (2 \times 1) = 17$ |

| 3 | 1 | |
|---|---|---|
| 1 | 2 | $\rightarrow (1 \times 3) + (2 \times 1) = 5$ |

| 81 | 27 | 9 | 3 | 1 | |
|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 1 | $\rightarrow (1 \times 81) + (0 \times 27) + (0 \times 9) + (3 \times 1) + (1 \times 1) = 85$ |

and $17 \times 5 = 85$, as expected.

# Exercises

1. Convert the following numbers from the base stated to base 10:

| | | | | | |
|---|---|---|---|---|---|
| (a) | 412 | base 5 | (b) | 333 | base 4 |
| (c) | 728 | base 9 | (d) | 1210 | base 3 |
| (e) | 1471 | base 8 | (f) | 612 | base 7 |
| (g) | 351 | base 6 | (h) | 111 | base 3 |

2. Convert the following numbers from base 10 to the base stated:

| | | | | | |
|---|---|---|---|---|---|
| (a) | 24 | to base 3 | (b) | 16 | to base 4 |
| (c) | 321 | to base 5 | (d) | 113 | to base 6 |
| (e) | 314 | to base 7 | (f) | 84 | to base 9 |
| (g) | 142 | to base 3 | (h) | 617 | to base 5 |

3. Carry out the following additions in the base stated:

| | | | | | |
|---|---|---|---|---|---|
| (a) | $3 + 2$ | in base 4 | (b) | $5 + 8$ | in base 9 |
| (c) | $4 + 6$ | in base 8 | (d) | $2 + 2$ | in base 3 |
| (e) | $6 + 7$ | in base 9 | (f) | $3 + 4$ | in base 6 |

4. In what number bases could each of the following numbers be written:

(a) 123 (b) 112 (c) 184

5. Carry out each of the following calculations in the base stated:

| | | | | | |
|---|---|---|---|---|---|
| (a) | $13 + 23$ | in base 4 | (b) | $120 + 314$ | in base 5 |
| (c) | $222 + 102$ | in base 3 | (d) | $310 + 132$ | in base 4 |
| (e) | $624 + 136$ | in base 7 | (f) | $211 + 142$ | in base 5 |
| (g) | $333 + 323$ | in base 4 | (h) | $141 + 424$ | in base 5 |

Check your answers to parts (a), (c) and (e) by converting to base 10 numbers.

6. Carry out each of the following multiplications in the base stated:

| | | | | | |
|---|---|---|---|---|---|
| (a) | $3 \times 2$ | in base 4 | (b) | $4 \times 3$ | in base 5 |
| (c) | $4 \times 2$ | in base 6 | (d) | $3 \times 5$ | in base 6 |
| (e) | $2 \times 2$ | in base 3 | (f) | $8 \times 8$ | in base 9 |

7. Carry out each of the following multiplications in the base stated:

(a) $121 \times 11$ in base 3 (b) $133 \times 12$ in base 4

(c) $13 \times 24$ in base 5 (d) $142 \times 14$ in base 5

(e) $161 \times 24$ in base 7 (f) $472 \times 32$ in base 8

(g) $414 \times 22$ in base 5 (h) $2101 \times 21$ in base 3

Check your answers to parts (a), (c) and (e) by converting to base 10 numbers.

8. In which base was each of the following calculations carried out?

(a) $4 + 2 = 11$ (b) $7 + 5 = 13$

(c) $8 \times 2 = 17$ (d) $4 \times 5 = 32$

(e) $11 - 3 = 5$ (f) $22 - 4 = 13$

9. (a) Change 147 in base 8 into a base 3 number.

(b) Change 321 in base 4 into a base 7 number.

(c) Change 172 in base 9 into a base 4 number.

(d) Change 324 in base 5 into a base 6 number.

10. In which base was each of the following calculations carried out?

(a) $171 \times 12 = 2272$ (b) $122 \times 21 = 11102$

(c) $24 \times 32 = 1252$ (d) $333 \times 33 = 23144$