STRAND D: SETS

Unit 10 *Logic and Venn Diagrams*

Student Text

Contents

10 Logic and Venn Diagrams

This unit introduces ideas of *logic*, a topic which is the foundation of all mathematics. We will be looking at logic puzzles and introducing some important work on sets.

10.1 Logic Puzzles

Here we introduce logic puzzles to help you think mathematically.

Example

Rachel, Tia and Millie are sisters. You need to deduce which sister is 9 years old, which one is 12 and which one is 14. You have two clues:

Clue 1 : *Tia's age is not in the 4-times table.*

Clue 2 : *Millie's age can be divided exactly by the number of days in a week.*

Solution

You can present this information in a logic table, shown opposite.

A *cross* in any box means that the statement is *not true*.

A *tick* in any box means that the statement *is true*.

	9 *yrs*	12 *yrs*	14 *yrs*
Rachel			
Tia			
Millie			

Clue 1 : *Tia's age is not in the 4-times table.*

This tells you that Tia's age is not 12.

Put a cross in Tia's row and column 12.

	9 *yrs*	12 *yrs*	14 *yrs*
Rachel			
Tia		✗	
Millie			

Clue 2 : *Millie's age can be divided exactly by the number of days in a week.*

This tells you that Millie's age is 14.

Put 2 crosses and a tick in Millie's row.

	9 *yrs*	12 *yrs*	14 *yrs*
Rachel			
Tia		✗	
Millie	✗	✗	✓

Looking at column '12 yrs', you can see that Rachel must be 12.

Fill in the ticks and crosses in Rachel's row.

	9 *yrs*	12 *yrs*	14 *yrs*
Rachel	✕	✓	✕
Tia		✕	
Millie	✕	✕	✓

Looking at column '9 yrs', you can see that Tia must be 9.

Tia's row can now be completed.

	9 *yrs*	12 *yrs*	14 *yrs*
Rachel	✕	✓	✕
Tia	✓	✕	✕
Millie	✕	✕	✓

Answer : Tia is 9 years old.

Rachel is 12 years old.

Millie is 14 years old.

Exercises

1. Jade, Billy and Kate each have one pet. They all own different types of pet.

   Clue 1: Kate's pet does not have a beak.

   Clue 2: Billy's pet lives in a kennel.

   Use this logic table to find out which pet each person owns.

	Dog	*Cat*	*Bird*
Jade			
Billy			
Kate			

2. Katie, John and Jodie each have a favourite sport: swimming, tennis or football. Use these clues to decide who likes which sport.

   Clue 1: In John's sport, a ball is hit with a racket.

   Clue 2: In Katie's sport a ball is kicked.

	Swimming	*Tennis*	*Football*
Katie			
John			
Jodie			

3. A waiter brings these meals to the table in a restaurant.

   Fried chicken with rice

   Baked fish with rice and beans

   Curry goat with potato fries and peas

   Use the clues to decide who eats which meal.

   - *Carl does not eat rice*
   - *Adam does not eat meat.*

4. Amanda, Joe, Alex and Zoe each have different coloured cars. One car is red, one blue, one white and the other is black.

   Decide which person has which coloured car.

   - *Amanda's car is not red or white.*
   - *Joe's car is not blue or white.*
   - *Alex's car is not black or blue.*
   - *Zoe's car is red.*

	Red	*Blue*	*White*	*Black*
Amanda				
Joe				
Alex				
Zoe				

5. Billy, John, Finlay and Jim are married to one of Mrs Brown, Mrs Green, Mrs Black and Mrs White.

   Use these clues and the table to decide who is married to who.

	Billy	*John*	*Finlay*	*Jim*
Mrs Brown				
Mrs Green				
Mrs Black				
Mrs White				

   Clues

   - *Mrs Brown's husband's first name does not begin with J.*
   - *Mrs Black's husband has a first name which has the same letter twice.*
   - *The first name of Mrs White's husband has 3 letters.*

6. In a race the four fastest runners were Alice, Leah, Nadida and Anna.

   Decide who finished in 1st, 2nd, 3rd and 4th places.

   - *Alice finished before Anna.*
   - *Leah finished before Nadida.*
   - *Nadida finished before Alice.*

10.2 Two Way Tables

Here we extend the ideas of the first section and present data in two way tables, from which we can either complete the tables or deduce information.

Example

Emma collected information about the cats and dogs that students in her class have. She filled in the table below, but missed out one number.

	Has a dog	*Does not have a dog*
Has a cat	8	4
Does not have a cat	12	

(a) Explain how to find the missing number if there are 30 students in Emma's class.

(b) How many students own at least one of these pets?

(c) Do more students own cats rather than dogs?

(d) Could it be true that some of the students do not have any pets?

Solution

(a) As there are 30 students in the class, each one has one entry in the complete table. As there are already

$$8 + 4 + 12 = 24$$

entries, the missing number is

$$30 - 24 = 6$$

	Has a dog	*Does not have a dog*
Has a cat	8	4
Does not have a cat	12	?

(b) All the students, except those in the bottom right hand square, own at least one cat or dog.

	Has a dog	*Does not have a dog*
Has a cat	8	4
Does not have a cat	12	6

Hence,

number of students owning at least one cat or dog is

$$30 - 6 = 24$$

(c) The total number of students owning a dog is given in the first column,

i.e. $8 + 12 = 20$

The total number of students owning a cat is given in the first row,

i.e. $8 + 4 = 12$

	Has a dog	*Does not have a dog*
Has a cat	8	4
Does not have a cat	12	6

So the answer to the question is NO, since there are more dog owners than cat owners.

(d) There are 6 students that do not own either a cat or a dog, but they might own another type of animal, etc., so we cannot deduce that some students have no pets.

Exercises

1. People leaving a football match were asked if they supported York United or Rivoli United. They were also asked if they were happy. The table below gives the results.

	York United FC	*Rivoli United FC*
Happy	40	8
Not happy	2	20

(a) How many York United supporters were happy?

(b) How many York United supporters were asked the questions?

(c) How many Rivoli United supporters were not happy?

(d) How many people were asked the questions?

(e) Which team do you think won the football match? What are your reasons for your answer?

2. The students in a class conducted a survey to find out how many students had videos at home and how many had computers at home. Their results are given in the table.

	Video	*No Video*
Computer	8	2
No Computer	20	3

(a) How many students did *not* have a video at home?

(b) How many students had a computer at home?

(c) How many students did *not* have a computer or a video at home?

(d) How many students were in the class?

3. 40 girls chose items for their student dorm rooms. Details of their colour choices are given below.

	Blue	*Orange*	*Purple*
Bedding	2	0	0
Towels	8		0
Kitchen items	1	8	0
Lamps	0	0	16

(a) How many girls chose orange towels?

(b) How many girls had blue as one of their choices?

(c) Which colour was the most popular?

4. During one month, exactly half of the 180 babies born in a hospital were boys, and 40 of the babies weighed 4 kg or more. There were 26 baby boys who weighed 4 kg or more.

	Less than 4 kg	*4 kg or more*
Boys		
Girls		

(a) Copy and complete the table above.

(b) How many baby girls weighed less than 4 kg when they were born?

5. In a survey about their animals, 18 families had only goats and 22 families had only cows. In the same survey, 16 of the families had caged birds and 24 did not have caged birds.

(a) How many families took part in the survey?

(b) Explain why it is impossible to complete the table below.

	Only goats	*Only cows*
Caged birds		
No caged birds		

(c) Complete the table if $\frac{3}{4}$ of the families with caged birds had only goats.

(d) How many families did *not* have caged birds and did *not* have cows?

10.3 Sets and Venn Diagrams

We use the idea of *sets* to classify numbers and objects and we use *Venn diagrams* to illustrate these sets.

Example

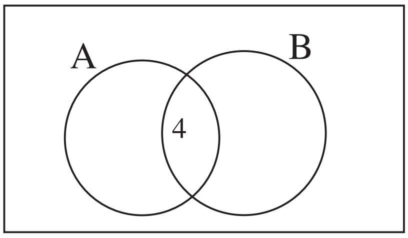
The sets A and B consist of numbers taken from the numbers 0, 1, 2, 3, . . . , 9 so that

Set A = { 4, 7, 9 }

Set B = { 1, 2, 3, 4, 5 }

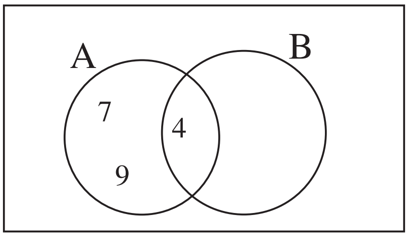
Illustrate these sets in a Venn diagram.

Solution

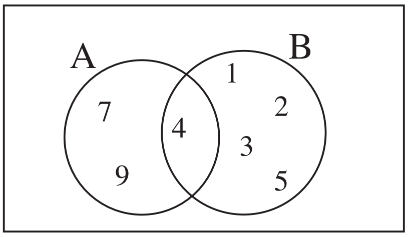


The framework for a Venn diagram is shown opposite, with the sets A and B indicated by the circles.

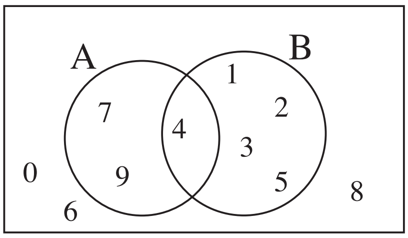
Since 4 is in both sets, it must be placed in the *intersection* of the two sets.



To complete set A, you put 7 and 9 in the part that does not intersect with B.



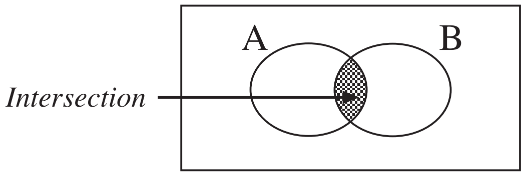
Similarly for B, you put 1, 2, 3 and 5 in the part that does not intersect with A.



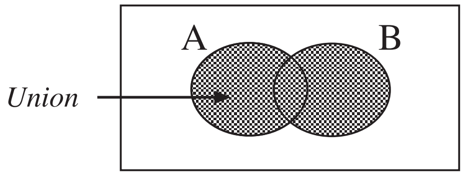
Finally, since the numbers 0, 6 and 8 have not been used in A or B, they are placed outside both A and B.

Note

The *intersection* of two sets consists of any numbers (or objects) that are in both A and B.



The *union* of two sets consists of any numbers (or objects) that are in A or in B or in both.



In the example above,

the intersection of A and B = { 4 }

the union of A and B = { 1, 2, 3, 4, 5, 7, 9 }

Note that, although the number 4 occurs in both A and B, it is *not* repeated when writing down the numbers in the union.

The *complement* of a set consists of any numbers (or objects) that are not in that set. In the example above,

the complement of A = { 0, 1, 2, 3, 5, 6, 8 }

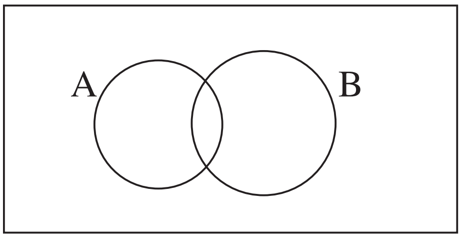
the complement of B = { 0, 6, 7, 8, 9 }

Exercises

1. Set A = { 1, 4, 5, 7, 8 }

   Set B = { 2, 6, 8, 10 }

   (a) Copy and complete the Venn diagram. Include all the whole numbers from 1 to 10.



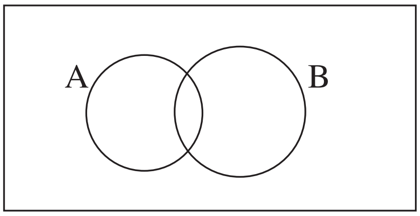
   (b) What is the intersection of A and B?

2. The whole numbers 1 to 10 are organised into 2 sets, set A and set B.

   Set A contains all the odd numbers.

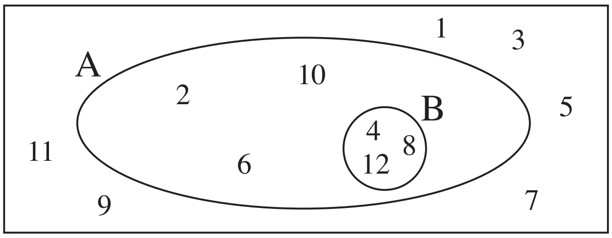
   Set B contains all the numbers greater than 4.

   (a) Copy and complete this diagram.



   (b) What is the union of A and B?

3. The whole numbers 1 to 12 are included in the Venn diagram.



   (a) List set A.

   (b) List set B.

   (c) Describe both sets in words.

   (d) What is the complement of A?

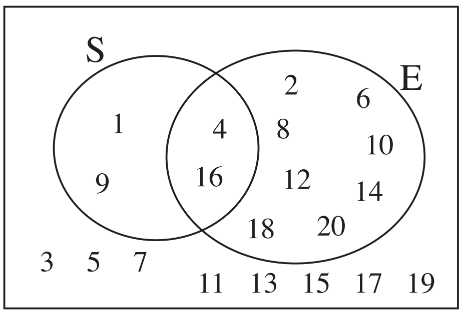
4. (a) Draw a Venn diagram to illustrate the sets P and Q. Include all the whole numbers from 1 to 15 in your diagram.

   P = { 3, 5, 7, 9 }

   Q = { 1, 3, 5, 7, 9, 11, 13, 15 }

   (b) What is the intersection of P and Q?

5. The whole numbers 1 to 20 are organised into sets as shown in the Venn diagram below.



(a) List set E.

(b) List set S.

(c) Describe each set in words.

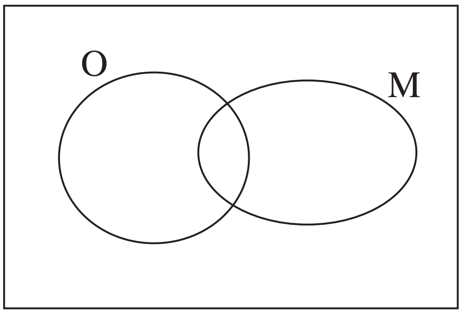
(d) What is the union of E and S?

6. The whole numbers 1 to 20 are organised into two sets,

O : Odd numbers

M : Multiples of 5

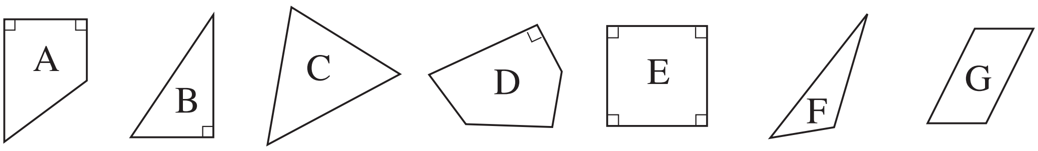
Copy and complete the Venn diagram, placing each number in the correct place.

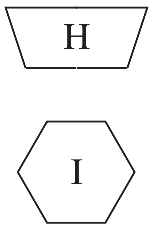


7. The shapes shown below are to be sorted into 2 sets, R and Q.

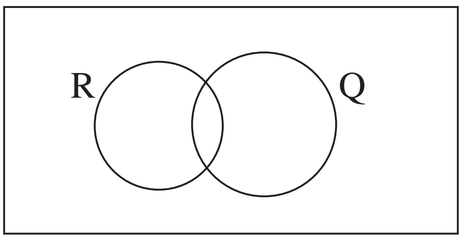
R contains shapes with a right angle.

Q contains shapes with four sides.





(a) Sort the shapes using the Venn diagram below.



(b) Which shapes are in both sets?

(c) Which shapes are in R but not in Q?

(d) Which shapes are not in R or Q?

8. Set P contains the letters needed to spell 'JENNY'.

   Set Q contains the letters needed to spell 'JEN'.

   Set R contains the letters needed to spell 'TED'.

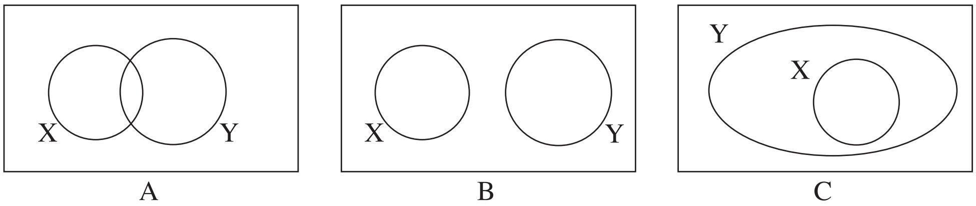
   (a) Draw a Venn diagram for the two sets, P and R.

   (b) Draw a Venn diagram for the two sets, P and Q.

   (c) What is the union of P and R?

   (d) What is the intersection of P and R?

9. Which of these Venn diagrams could be used to illustrate the sets described below?



   (a) X is the set of all squares.
   Y is the set of all rectangles.

   (b) X is the set of all triangles.
   Y is the set of all squares.

   (c) X is the set of all quadrilaterals (4-sided shapes).
   Y is the set of all triangles.

   (d) X is the set of all shapes containing a right angle.
   Y is the set of all triangles.

10.4 Set Notation

We use U to denote the *universal* set, that is, the set from which we are picking the members of A, B, ...

$A \cap B$, the intersection of A and B, is the set of members in set A *and* in set B.

$A \cup B$, the union of A and B, is the set of members in set A or in set B or in both.

A', the complement of A, is the set of members in U but not in A.

$A \subset B$ means that A is a subset of B, i.e. every element in A is also in B.

$\varnothing$ is the empty set, i.e. the set with no numbers (or objects).

$n(A)$ is the number of members in set A.

Example 1

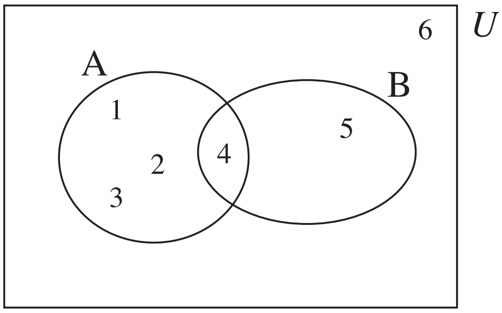
If $U = \{1, 2, 3, 4, 5, 6\}$ and $A = \{1, 2, 3, 4\}$, $B = \{4, 5\}$

find (a) $A \cap B$, (b) $A \cup B$ (c) A' (d) B'

Is $B \subset A$?

Solution

First put the numbers in a Venn diagram.



(a) $A \cap B = \{4\}$

(b) $A \cup B = \{1, 2, 3, 4, 5\}$

(c) $A' = \{5, 6\}$

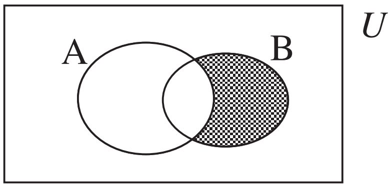
(d) $B' = \{1, 2, 3, 6\}$

No, B is not a subset of A since the number 5 is in B but not in A.

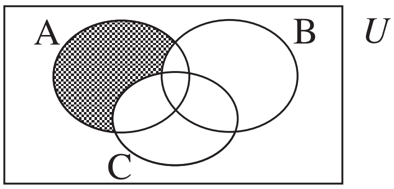
Example 2

Use set notation to describe the shaded regions of these diagrams.

(a)



(b)



Solution

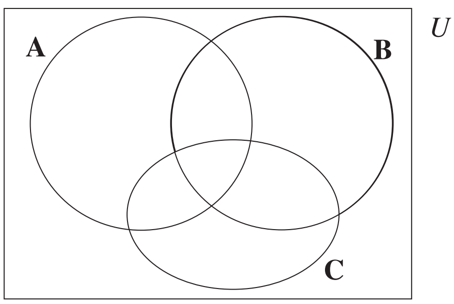
(a) This is the intersection of B with A', i.e. $B \cap A'$.

(b) This is the intersection of A with the complement of the union of B and C, i.e. $A \cap (B \cup C)'$.

Example 3

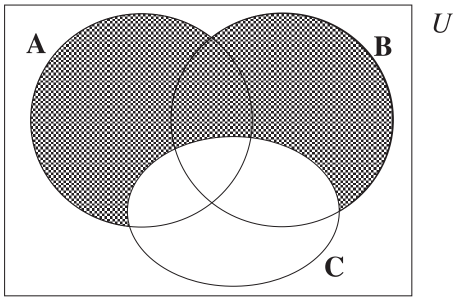
On this diagram, shade the region that represents

$(A \cup B) \cap C'$



Solution

You want the union of A and B which is not in C.



Exercises

1. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

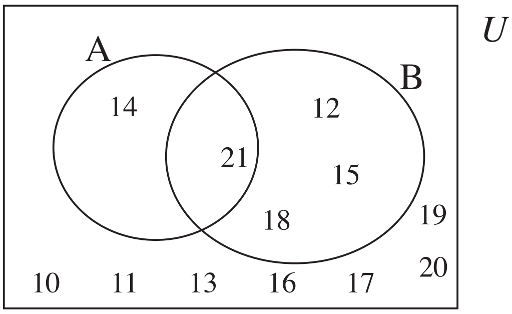
   $A = \{2, 4, 6, 8\}$

   and $B = \{3, 6, 9\}$

   find:

(a)	$A \cap B$	(b)	$A \cup B$	(c)	A'
(d)	B'	(e)	$A' \cap B'$	(f)	$A' \cup B'$

2. The Venn diagram illustrates sets A, B and U.

   Find:

(a)	$A \cap B$	(b)	$(A \cap B)'$
(c)	$A \cup B$	(d)	A'
(e)	B'	(f)	$A' \cap B'$
(g)	$A' \cap B$		

3. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

   $A = \{1, 3, 6, 10\}$

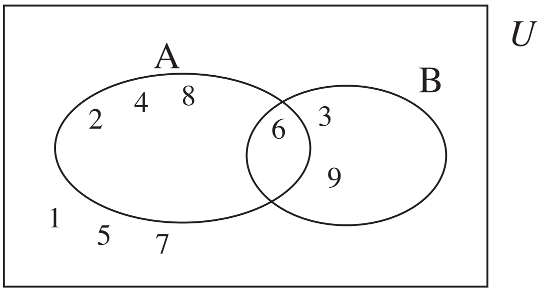
   $B = \{1, 5, 10\}$

   and $C = \{3, 6, 9, 12\}$,

   find:

(a)	$A \cap B$	(b)	$A \cap C$	(c)	$B \cap C$
(d)	$A \cup B$	(e)	$A \cup C$	(f)	C'
(g)	$A \cap C'$	(h)	B'	(i)	$B' \cup C'$
(j)	$A \cap B \cap C$	(k)	$A \cup B \cup C$		

4.



In the Venn diagram above,

$U = \{\text{positive whole numbers less than } 10\}$, and

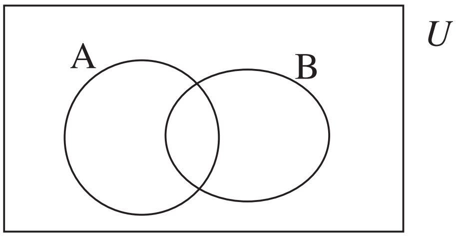
A and B are subsets of U.

(i) Describe A and B in words.

(ii) List the members of $A \cap B$ and describe the set, in words, in relation to A and B.

(iii) Determine $n(A \cup B)'$.

5. Make a separate copy of this diagram for each part of the question.



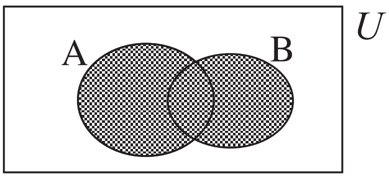
Shade the region on the diagram that represents:

(a) $A \cap B$ (b) A' (c) $A \cup B'$

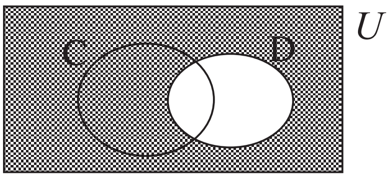
(d) $A' \cap B'$ (e) $A \cap B'$ (f) $(A \cup B)'$

6. Use set notation to describe the region shaded in each of these diagrams.

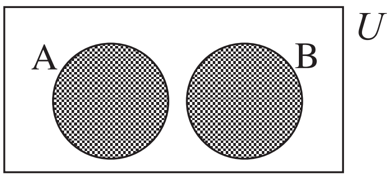
(a)



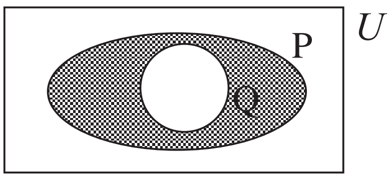
(b)

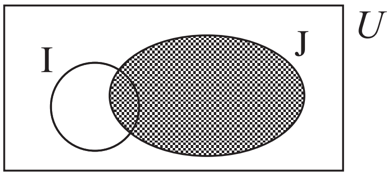


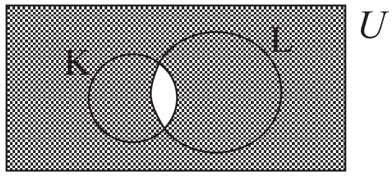
(c)



(d)

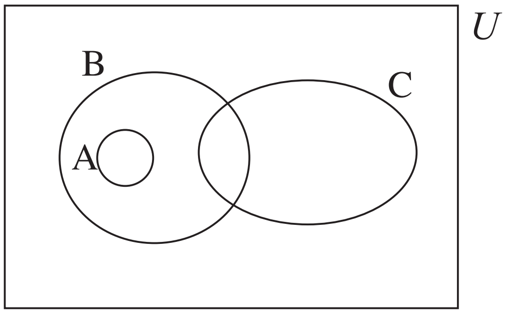


(e) 

(f) 

7. The diagram illustrates 3 sets, A, B and C.

Say whether each of these statements is *true* or *false*.



(a) $A \subset B$

(b) $B \subset C$

(c) $A \cap B = A$

(d) $A \cap B = C$

(e) $A \cap C = \varnothing$

(f) $B \cap C = \varnothing$

(g) $B \cup A = B$

8. If $U = \{a, b, c, d, e, f, g, h\}$

$A = \{a, c, e\}$

$B = \{b, d, g, h\}$

and $C = \{a, c, e, f\}$,

say whether each of these statements is *true* or *false*. Write correct statements to replace those that are false.

(a) $B \cap C = \varnothing$

(b) $C \subset A$

(c) $B \cup C = U$

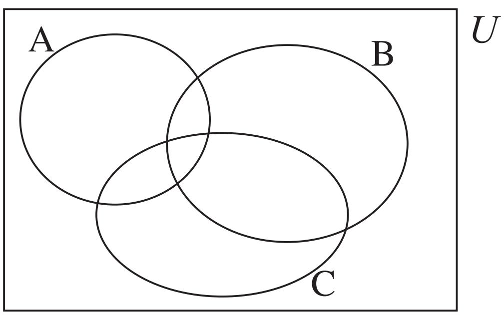
(d) $A \cap C = \{a, c, e, f\}$

(e) $(A \cap C)' = \{b, d, f, g, h\}$

(f) $B \subset U$

(g) $A \cap B' = C$

9. For each part of the question, use a copy of the diagram.



Shade the region of the diagram that represents:

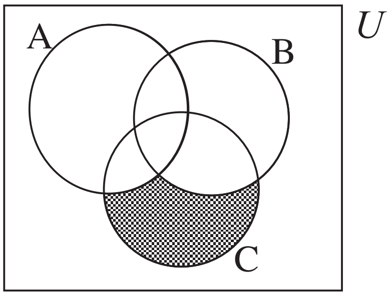
(a) $A \cap B \cap C$ (b) $(A \cup B) \cap C$

(c) $(A \cap B) \cup C$ (d) $A' \cap (B \cup C)$

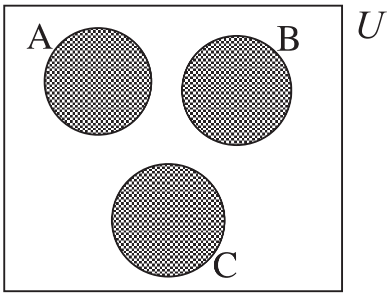
(e) $A' \cap B \cap C$ (f) $A' \cap B' \cap C'$

10. Use set notation to describe the regions shaded in each of these diagrams.

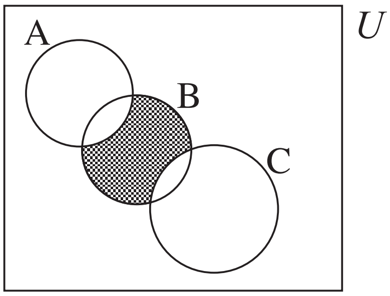
(a)



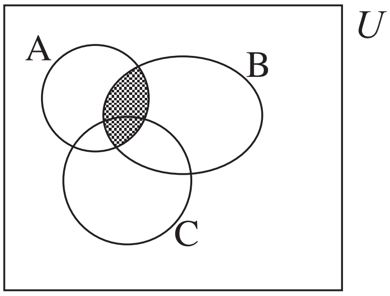
(b)



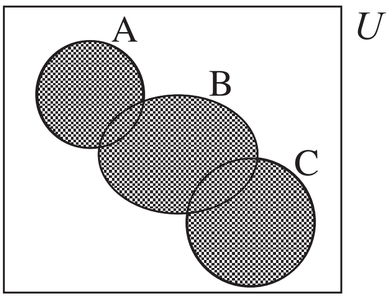
(c)



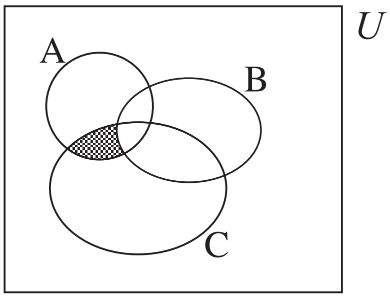
(d)



(e)



(f)



10.5 Logic Problems and Venn Diagrams

Venn diagrams can be very helpful in solving logic problems.

Example 1

In a class there are

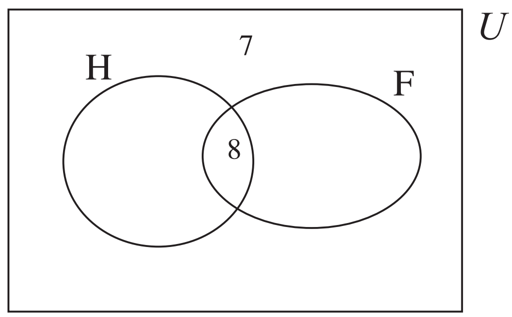
- 8 students who play football and cricket
- 7 students who do not play football or cricket
- 13 students who play cricket
- 19 students who play football.

How many students are there in the class?

Solution

You can use a Venn diagram to show the information.

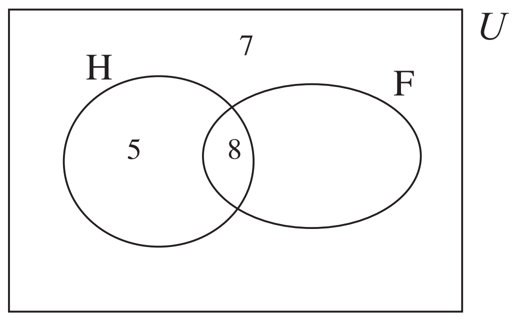
The first two sets of students can be put directly on to the diagram.



If there are 13 students who play cricket, and we already know that 8 play cricket and football, then there must be

$$13 - 8 = 5$$

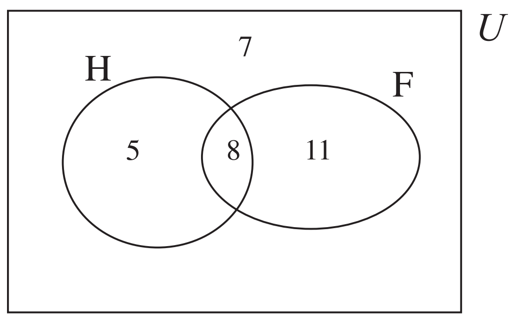
who play just cricket.



Similarly for football,

$$19 - 8 = 11$$

play just football.



So the total number of students in the class is

$$7 + 5 + 8 + 11 = 31$$

Example 2

A survey of 156 visitors to the Caribbean found that:

118 persons visited Barbados

98 persons visited Antigua

110 persons visited Tobago

25 persons visited Barbados and Antigua only

35 persons visited Barbados and Tobago only

30 persons visited Tobago and Antigua only

x visitors visited all three countries.

Every visitor visited at least one of the three islands.

(a) Draw a carefully labelled Venn diagram to represent the information above.

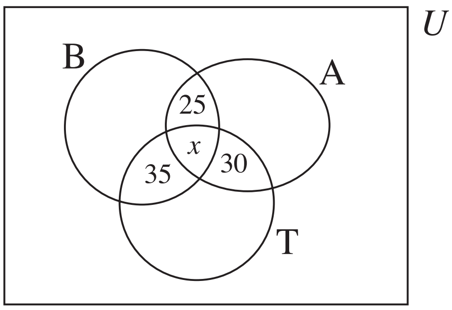
(b) Write an algebraic expression in x to represent the number of travellers who visited Barbados only.

(c) Write an equation in x to show the total number of visitors in the survey.

(d) Calculate the number of travellers who visited all three countries.

Solution

(a)



(b) Number who visited Barbados only

$= 118 - (25 + x + 35)$

$= 118 - 60 - x$

$= 58 - x$

Number who visited Antigua only

$= 98 - (25 + x + 30)$

$= 98 - 55 - x$

$= 43 - x$

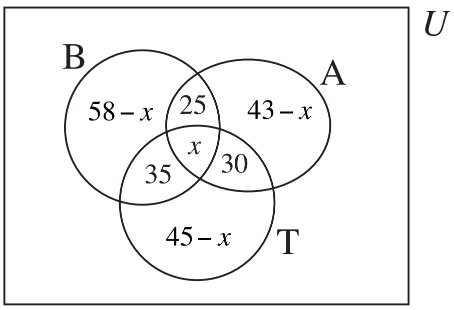
Number who visited Tobago only

$= 110 - (35 + x + 30)$

$= 110 - 65 - x$

$= 45 - x$

(c)



Total number of visitors $= 156$

$= (58 - x) + (43 - x) + (45 - x) + 25 + 30 + 35 + x$

i.e. $156 = 236 - 2x$

$2x = 236 - 156$

$= 80$

$x = 40$

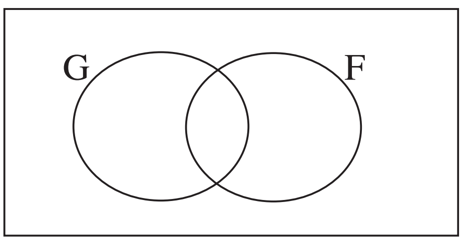
(d) 40 travellers visited all three countries.

Exercises

1. In a group of six friends everybody plays football or basketball. 4 members of the group play both sports and 1 member of the group plays only basketball. How many play only football?

2. John's mum buys 5 hamburgers. All the hamburgers have ketchup or hot sauce on them. Some have ketchup and hot sauce. There are 2 hamburgers with ketchup and hot sauce and one hamburger with only hot sauce. How many hamburgers have only ketchup on them?

3. This diagram represents a class of students. G is the set of girls and F is the set of students who like football. Make 4 copies of this diagram.



On separate diagrams, shade the part that represents:

(a) girls who like football, (b) girls who dislike football,

(c) boys who like football, (d) boys who do not like football.

4. In a class of 32 pupils, 20 say that they like pancakes and 14 say that they like maple syrup. There are 6 pupils who do not like either. How many of them like both pancakes and maple syrup?

5. On a garage forecourt there are 6 new cars, 12 red cars and no others.

   (a) What is the maximum possible number of cars on the forecourt?

   (b) What is the smallest possible number of cars on the forecourt?

   (c) If 2 of the new cars are red, how many cars are on the forecourt?

6. A pencil case contains 20 pens that are red or blue. Of these, 8 are blue and 6 do not work. How many of the blue pens do not work if there are 8 red pens that do work?

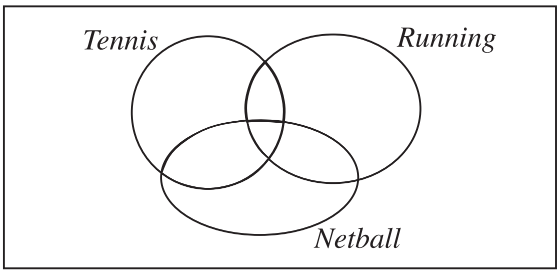
7. In a school canteen there are 45 students. There are 16 who have finished eating. The others are eating either sausages or fries, or both sausages and fries. There are 26 eating fries and 17 eating sausages.

   (a) How many are eating sausages and fries?

   (b) How many are eating sausages without fries?

   (c) How many are eating only fries?

8. Youth club members can choose tennis, running or netball. The diagram below represents the possible combinations.



Make 3 copies of the diagram.

On separate diagrams, shade the parts that represent:

   (a) those who take part in all three sports,

   (b) those who play tennis and netball, but not running,

   (c) those who play only tennis.

9. All the members of a group of 30 teenagers belong to at least one club. There are 3 clubs: chess, drama and art.

   6 of the teenagers belong to only the art club.

   5 of the teenagers belong to all 3 clubs.

   2 of the teenagers belong to the chess and art clubs but not to the drama club.

   15 of the teenagers belong to the art club.

   2 of the teenagers belong only to the chess club.

   3 of the teenagers belong only to the drama club.

   (a) How many of the group belong to the chess club and the drama club, but not the art club?

   (b) How many of the group belong to each club?