STRAND J: Vectors and Matrices

Unit 38 *Vectors*

Student Text

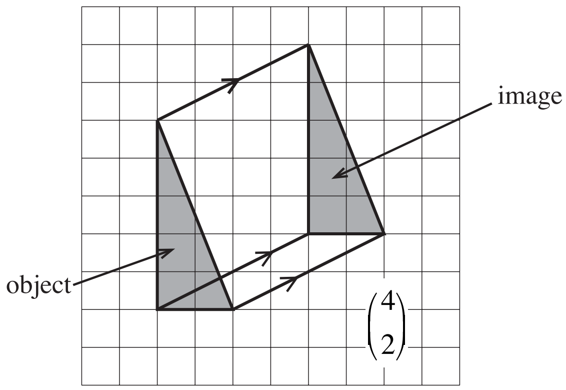
Contents

Section

38 Vectors

38.1 Vectors and Scalars

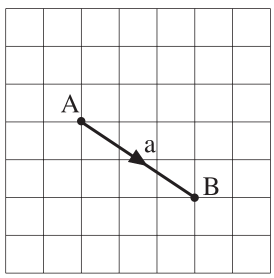
Vectors are used in Unit 36 to describe translations. The diagram shows the translation of a triangle by the vector $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$.



Note that the vector $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ specifies *how far* the triangle is to be moved and the *direction*, i.e. 4 units *horizontally* (to the right) and 2 units *vertically* (up).

All vectors have **length** (or size) and **direction**. Quantities which do not have direction, but only length or size are known as *scalar quantities*. Quantities like *mass, length, area* and *speed* are *scalars* because they have size only, while quantities like *force* and *velocity* are *vectors* because they have a direction as well as a size.

The two points A and B are shown in the diagram. The displacement (change of position) of B from A is a vector because it has length and a direction.



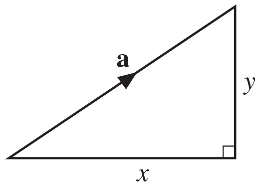
We can write this displacement as $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ or label the vector **a** and write

$\mathbf{a} = \overrightarrow{AB}$ or $\mathbf{a} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and in this format, it is called a *column vector.*

The notation **a** is used when **a** is a vector and the notation *a* is used when *a* is a scalar.

The length of a vector is called its **magnitude** or **modulus**: we write this as $|\mathbf{a}|$.

If $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$, then, using Pythagoras' Theorem,

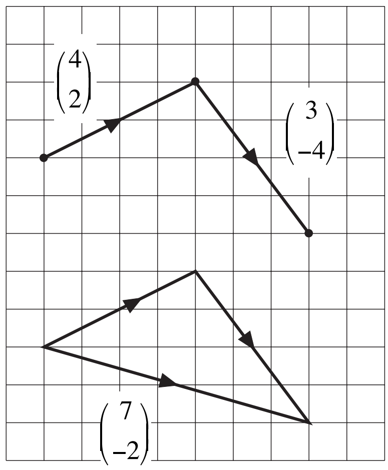


$$|\mathbf{a}| = \sqrt{x^2 + y^2}$$

So for the vector $\mathbf{a} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $|\mathbf{a}| = \sqrt{3^2 + (-2)^2} = \sqrt{9 + 4} = \sqrt{13}$

Vectors can simply be added and subtracted.

Consider $\begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ which can be represented as shown in the following diagram.



So, from the diagram, the *addition* of these two vectors can be written as a single vector $\begin{pmatrix} 7 \\ -2 \end{pmatrix}$, which is just the addition of each component of the original vector. In general,

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a + c \\ b + d \end{pmatrix}$$

A similar result is true for *subtraction*,

$$\begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a - c \\ b - d \end{pmatrix}$$

A vector can be *multiplied* by a scalar, i.e. a number, by multiplying each component by that scalar.

For example, $4 \times \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 12 \\ -8 \end{pmatrix}$.

In general,

$$k \times \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix}$$

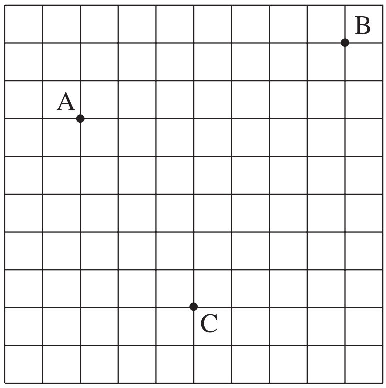
Worked Example 1

Write each of the following vectors in the form $\begin{pmatrix} a \\ b \end{pmatrix}$.

(a) $\overrightarrow{AB}$

(b) $\overrightarrow{BC}$

(c) $\overrightarrow{AC}$



Solution

From the diagram:

(a) $\overrightarrow{AB} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$

(b) $\overrightarrow{BC} = \begin{pmatrix} -4 \\ -7 \end{pmatrix}$

(c) $\overrightarrow{AC} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$

Note

In the Worked Example above, we see that

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

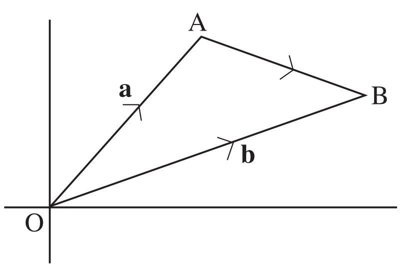
This is always true so that, for example,

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

or

$$\mathbf{a} + \overrightarrow{AB} = \mathbf{b}$$

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$



Similarly,

$$\overrightarrow{BA} = \mathbf{a} - \mathbf{b}$$

as the direction is the opposite of $\overrightarrow{AB}$.

Worked Example 2

If $\mathbf{a} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$ find:

(a) $2\mathbf{a}$ (b) $\mathbf{b} + \mathbf{c}$ (c) $\mathbf{a} - \mathbf{b}$ (d) $2\mathbf{a} + 3\mathbf{b}$

Solution

(a) $2\mathbf{a} = 2 \times \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \times 3 \\ 2 \times 7 \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \end{pmatrix}$

(b) $\mathbf{b} + \mathbf{c} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} + \begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 + (-3) \\ -4 + (-4) \end{pmatrix} = \begin{pmatrix} -1 \\ -8 \end{pmatrix}$

(c) $\mathbf{a} - \mathbf{b} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 - 2 \\ 7 - (-4) \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \end{pmatrix}$

(d) $2\mathbf{a} + 3\mathbf{b} = 2 \times \begin{pmatrix} 3 \\ 7 \end{pmatrix} + 3 \times \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \times 3 \\ 2 \times 7 \end{pmatrix} + \begin{pmatrix} 3 \times 2 \\ 3 \times (-4) \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \end{pmatrix} + \begin{pmatrix} 6 \\ -12 \end{pmatrix}$

$$= \begin{pmatrix} 6 + 6 \\ 14 + (-12) \end{pmatrix} = \begin{pmatrix} 12 \\ 2 \end{pmatrix}$$

Note

When the vector starts at the origin, it is called a *position vector.*

Worked Example 3

$\overrightarrow{OA}$ and $\overrightarrow{OB}$ are position vectors relative to the origin, O. Given the points A (3, 1) and B $(-1, -2)$

(a) write down $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as column vectors

(b) express $\overrightarrow{AB}$ as a column vector

(c) calculate the length of $\overrightarrow{AB}$.

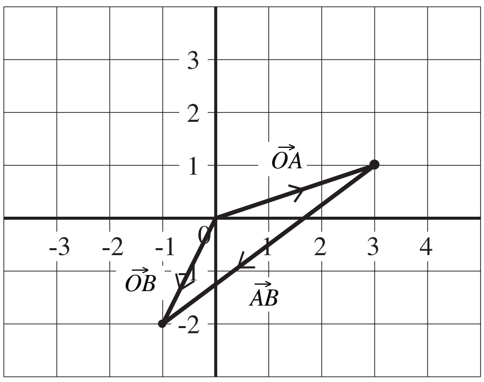
Solution

(a) $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

(b) As $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$$

This can be seen in the diagram opposite.



(c) $\left|\vec{OB}\right| = \sqrt{(-4)^2 + (-3)^2}$

$= \sqrt{16 + 9}$

$= \sqrt{25}$

$= 5$

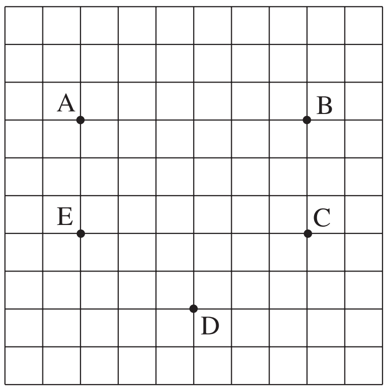
Exercises

1. Which of the following are vectors and which are scalars:

   (a) Time (b) Velocity (c) Speed

   (d) Force (e) Distance (f) Temperature

2. Use the points in the grid below to write the vectors given in column vector form.



   (a) $\vec{AB}$ (b) $\vec{AC}$

   (c) $\vec{DE}$ (d) $\vec{BE}$

   (e) $\vec{EB}$ (f) $\vec{AD}$

   (g) $\vec{CD}$ (h) $\vec{DC}$

   What is the relationship between

   $\vec{AC}$ and $\vec{CA}$?

3. Plot the positions of the points A, B, C, D, E and F relative to a point O if:

   (a) $\vec{OA} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ (b) $\vec{OB} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ (c) $\vec{AC} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

   (d) $\vec{BD} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$ (e) $\vec{CE} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$ (f) $\vec{DF} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

   Write the vector $\vec{EF}$ as a column vector.

4. If $\mathbf{a} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$, find:

   (a) $\mathbf{a} + \mathbf{b}$ (b) $\mathbf{b} + \mathbf{c}$ (c) $\mathbf{a} + \mathbf{c}$

   (d) $\mathbf{a} - \mathbf{b}$ (e) $\mathbf{b} - \mathbf{a}$ (f) $\mathbf{c} - \mathbf{a}$

   (g) $3\mathbf{a}$ (h) $-2\mathbf{b}$ (i) $4\mathbf{c}$

   (j) $2\mathbf{a} + 3\mathbf{b}$ (k) $5\mathbf{c} - 3\mathbf{a}$ (l) $4\mathbf{b} - 2\mathbf{c}$

5. If $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$, find:

(a) $3\mathbf{a} + 2\mathbf{b}$ (b) $4\mathbf{a} + 3\mathbf{c}$ (c) $6\mathbf{a} - 3\mathbf{b}$

(d) $4\mathbf{c} + 2\mathbf{a}$ (e) $3\mathbf{c} - 2\mathbf{a}$ (f) $3\mathbf{a} + 2\mathbf{b} - 5\mathbf{c}$

6. If $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$, solve the equations below to find the column vector **x**.

(a) $\mathbf{a} + \mathbf{x} = \mathbf{b}$ (b) $\mathbf{x} - \mathbf{c} = \mathbf{a}$ (c) $\mathbf{x} + \mathbf{b} = \mathbf{c}$

(d) $2\mathbf{x} + \mathbf{a} = \mathbf{b}$ (e) $3\mathbf{a} + 2\mathbf{x} = \mathbf{c}$ (f) $4\mathbf{a} - \mathbf{x} = \mathbf{c}$

(g) $3\mathbf{a} + 2\mathbf{x} = 4\mathbf{b}$ (h) $\mathbf{a} - 2\mathbf{x} = 4\mathbf{c}$ (i) $3\mathbf{b} + 2\mathbf{x} = \mathbf{c}$

7. In this question $\mathbf{a} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

For each part draw the vectors listed on separate diagrams.

(a) $\mathbf{a},\ \mathbf{b},\ \mathbf{a} + \mathbf{b}$ (b) $\mathbf{a},\ \mathbf{c},\ \mathbf{a} - \mathbf{c}$ (c) $\mathbf{b},\ 2\mathbf{b},\ 3\mathbf{b}$

(d) $\mathbf{c},\ -\mathbf{c},\ -2\mathbf{c}$ (e) $\mathbf{a},\ \mathbf{b},\ 2\mathbf{a} - 3\mathbf{b}$

8. The vertices of a quadrilateral, OPQR are (0, 0), (4, 2), (6, 10) and (2, 8) respectively.

(a) **Using a vector method**, express in the form $\begin{pmatrix} x \\ y \end{pmatrix}$, the vector

(i) $\overrightarrow{OP}$

(ii) $\overrightarrow{RQ}$

(b) Calculate $\left|\overrightarrow{OP}\right|$, the magnitude of $\overrightarrow{OP}$.

(c) State TWO geometrical relationships between the line segments OP and RQ.

(CXC)

9. Given that $\overrightarrow{PR} = \begin{pmatrix} b \\ -2b \end{pmatrix}$

and $\overrightarrow{PS} = \begin{pmatrix} 3b \\ b + 1 \end{pmatrix}$,

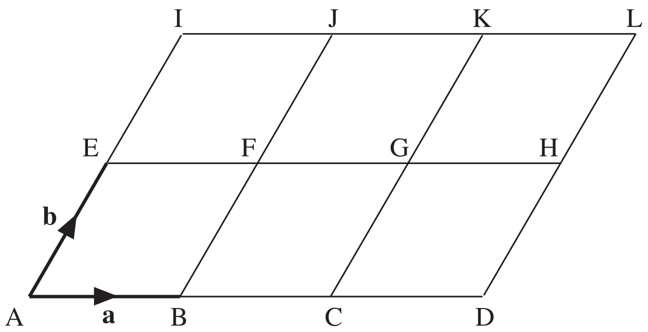
(a) express EACH of the vectors $\overrightarrow{RP}$ and $\overrightarrow{RS}$ in the simplest form

(b) determine the values of b if $\left|\overrightarrow{PR}\right| = \sqrt{20}$ units.

(CXC)

38.2 Vectors and Geometry

Vectors can be used to solve problems in geometry. In two dimensions, it is possible to describe the position of any point using two vectors. For example, using the vectors **a** and **b** shown in the diagram:



$\overrightarrow{AC} = 2\mathbf{a}$

$\overrightarrow{EF} = \mathbf{a}$

$\overrightarrow{IK} = 2\mathbf{a}$

$\overrightarrow{AF} = \mathbf{a} + \mathbf{b}$

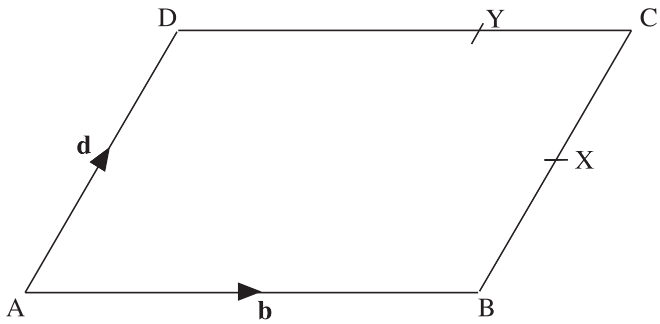
$\overrightarrow{AL} = 3\mathbf{a} + 2\mathbf{b}$

$\overrightarrow{LE} = -3\mathbf{a} - \mathbf{b}$

Note that $\overrightarrow{AB}$, $\overrightarrow{EF}$ and $\overrightarrow{IK}$ are all *parallel vectors* as they have the same direction.

Worked Example 1

In the parallelogram shown below, $\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{AD} = \mathbf{d}$. Also X is the midpoint of BC and Y lies on DC such that $DY = 2CY$.



Express the following vectors in terms of **b** and **d**.

(a) $\overrightarrow{AC}$ (b) $\overrightarrow{BX}$ (c) $\overrightarrow{AX}$

(d) $\overrightarrow{DY}$ (e) $\overrightarrow{AY}$ (f) $\overrightarrow{XY}$

Solution

(a) $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}$

$= \mathbf{b} + \mathbf{d}$

(b) $\overrightarrow{BX} = \frac{1}{2}\overrightarrow{BC}$

$= \frac{1}{2}\mathbf{d}$

(c) $\vec{AX} = \vec{AB} + \vec{BX}$

$= \mathbf{b} + \frac{1}{2}\mathbf{d}$

(d) $\vec{DY} = \frac{2}{3}\vec{DC}$

$= \frac{2}{3}\mathbf{b}$

(e) $\vec{AY} = \vec{AD} + \vec{DY}$

$= \mathbf{d} + \frac{2}{3}\mathbf{b}$

(f) $\vec{XY} = \vec{XA} + \vec{AY}$

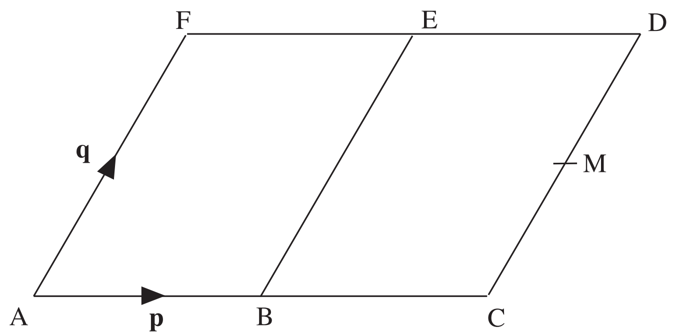
$= -\left(\mathbf{b} + \frac{1}{2}\mathbf{d}\right) + \left(\mathbf{d} + \frac{2}{3}\mathbf{b}\right)$

$= \frac{2}{3}\mathbf{b} - \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{d}$

$= -\frac{1}{3}\mathbf{b} + \frac{1}{2}\mathbf{d}$

Worked Example 2

The diagram shows 2 identical parallelograms. The vector $\mathbf{q} = \vec{AF}$ and the vector $\mathbf{p} = \vec{AB}$. The point M is the midpoint of CD.



(a) Show that BM is parallel to AD.

(b) Show that EM is parallel to FC.

Solution

(a) $\vec{BM} = \vec{BC} + \vec{CM}$

$= \mathbf{p} + \frac{1}{2}\mathbf{q}$

$$\vec{AD} = \vec{AC} + \vec{CD}$$
$$= 2\mathbf{p} + \mathbf{q}$$

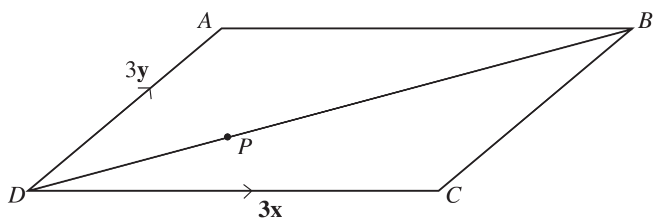
As $\vec{AD} = 2\vec{BM}$, the lines AD and BM must be parallel.

(b) $$\vec{EM} = \vec{ED} + \vec{DM}$$
$$= \mathbf{p} + \left(-\frac{1}{2}\mathbf{q}\right)$$
$$= \mathbf{p} - \frac{1}{2}\mathbf{q}$$

$$\vec{FC} = \vec{FD} + \vec{DC}$$
$$= 2\mathbf{p} - \mathbf{q}$$

As $\vec{FC} = 2\vec{EM}$, the lines FC and EM must be parallel.

Worked Example 3



In the figure above, **not drawn to scale**, *ABCD* is a parallelogram such that $\vec{DC} = 3\mathbf{x}$ and $\vec{DA} = 3\mathbf{y}$. The point *P* is on *DB* such that $DP : PB = 1:2$.

(a) Express in terms of **x** and **y**.

(i) $\vec{AB}$

(ii) $\vec{BD}$

(iii) $\vec{DP}$

(b) Show that $\vec{AP} = \mathbf{x} - 2\mathbf{y}$.

(c) Given that *E* is the mid-point of *DC*, prove that *A*, *P* and *E* are collinear.

(d) Given that $\mathbf{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, use a vector method to prove that triangle *AED* is isosceles.

(CXC)

Solution

(a) (i) $\vec{AB} = \vec{DC} = 3\mathbf{x}$

(ii) $\vec{BD} = \vec{BA} + \vec{AD} = -3\mathbf{x} - 3\mathbf{y} = -3(\mathbf{x} + \mathbf{y})$

(iii) $\vec{DP} = \frac{1}{3}\vec{DB} = -\frac{1}{3}\vec{BD} = -\frac{1}{3}(-3)(\mathbf{x} + \mathbf{y}) = \mathbf{x} + \mathbf{y}$

(b) $\vec{AP} = \vec{AD} + \vec{DP} = -3\mathbf{y} + (\mathbf{x} + \mathbf{y}) = \mathbf{x} - 2\mathbf{y}$

(c) $\vec{DE} = \frac{3}{2}\mathbf{x}$. so $\vec{PE} = \vec{PD} + \vec{DE} = -\mathbf{x} - \mathbf{y} + \frac{3}{2}\mathbf{x} = \frac{1}{2}\mathbf{x} - \mathbf{y}$

Hence

$$2 \times \vec{PE} = \mathbf{x} - 2\mathbf{y} = \vec{AP}$$

that is $\vec{PE}$ and $\vec{AP}$ have the same direction. So *A*, *P* and *E* are collinear.

(d) We need to find two sides of equal length.

Now

$$\vec{DE} = \frac{3}{2}\mathbf{x} = \frac{3}{2}\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix}$$

and

$$\left|\vec{DE}\right| = \sqrt{3^2 + 0^2} = \sqrt{9 + 0} = \sqrt{9} = 3$$

$$\vec{AE} = \vec{AD} + \vec{DE} = -3\mathbf{y} + \frac{3}{2}\mathbf{x} \quad = -3\begin{pmatrix}1\\1\end{pmatrix} + \frac{3}{2}\begin{pmatrix}2\\0\end{pmatrix}$$

$$= \begin{pmatrix}-3\\-3\end{pmatrix} + \begin{pmatrix}3\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\-3\end{pmatrix}$$

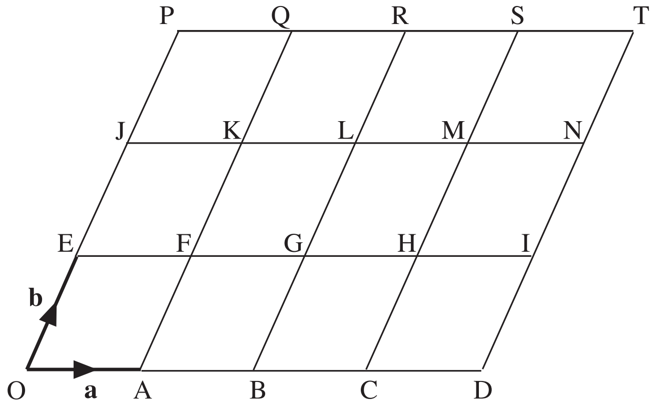
So

$$\left|\vec{AE}\right| = \sqrt{0^2 + (-3)^2} = \sqrt{0 + 9} = \sqrt{9} = 3$$

As $\left|\vec{AE}\right| = \left|\vec{DE}\right| = 3$, triangle *ADE* is isosceles.

Exercises

1. The diagram shows a grid made up of sets of equally spaced parallel lines. The vectors $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OE} = \mathbf{b}$ are shown on the grid.



Write each of the following vectors in terms of **a** and **b**.

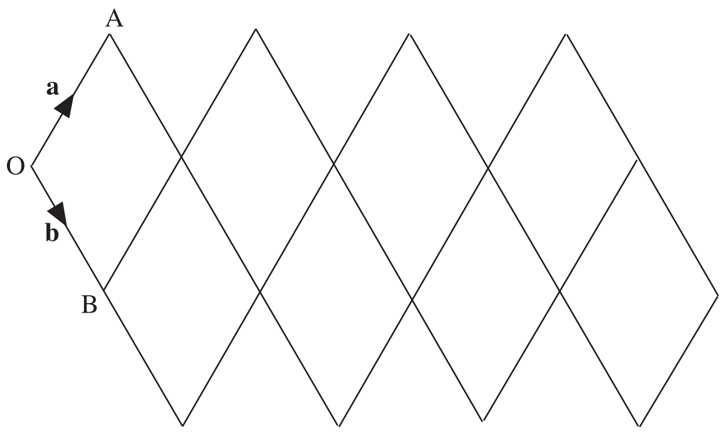
(a) $\overrightarrow{OD}$ (b) $\overrightarrow{AB}$ (c) $\overrightarrow{BG}$ (d) $\overrightarrow{IS}$

(e) $\overrightarrow{JP}$ (f) $\overrightarrow{ES}$ (g) $\overrightarrow{AQ}$ (h) $\overrightarrow{CS}$

(i) $\overrightarrow{PK}$ (j) $\overrightarrow{PG}$ (k) $\overrightarrow{RF}$ (l) $\overrightarrow{SE}$

(m) $\overrightarrow{CP}$ (n) $\overrightarrow{GE}$ (o) $\overrightarrow{IJ}$ (p) $\overrightarrow{TA}$

2. The diagram shows a grid made up of two sets of parallel lines. The vectors $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OE} = \mathbf{b}$ are shown on the grid.



Copy the grid and use the following information to label each point where lines meet.

$\overrightarrow{OC} = 2\mathbf{b}$ $\overrightarrow{OD} = \mathbf{a} + \mathbf{b}$ $\overrightarrow{DE} = \mathbf{a}$

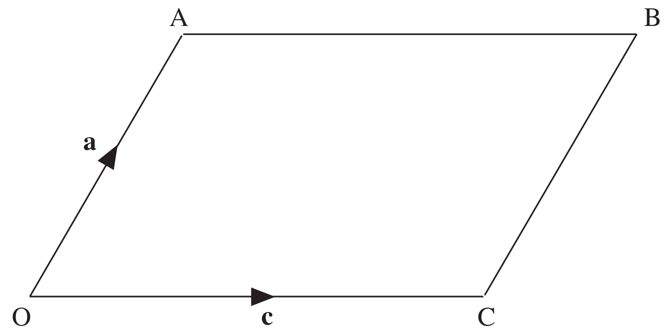
$\overrightarrow{EF} = \mathbf{b}$ $\overrightarrow{BG} = \mathbf{a} + \mathbf{b}$ $\overrightarrow{BH} = 2\mathbf{a} + 3\mathbf{b}$

$\overrightarrow{HI} = -\mathbf{b}$ $\overrightarrow{IJ} = \mathbf{a} - \mathbf{b}$ $\overrightarrow{JK} = 3\mathbf{b} + \mathbf{a}$

$\overrightarrow{KL} = -\mathbf{a}$ $\overrightarrow{OM} = 4\mathbf{a} + 3\mathbf{b}$ $\overrightarrow{MN} = -\mathbf{a}$

$\overrightarrow{BP} = 4\mathbf{a} + 3\mathbf{b}$ $\overrightarrow{BQ} = \mathbf{a} + 2\mathbf{b}$ $\overrightarrow{AR} = 2\mathbf{a} + 4\mathbf{b}$

3. The diagram shows the parallelogram OABC, in which $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.



(a) Write the following vectors in terms of **a** and **c**.

(i) $\overrightarrow{AB}$ (ii) $\overrightarrow{CB}$ (iii) $\overrightarrow{BC}$

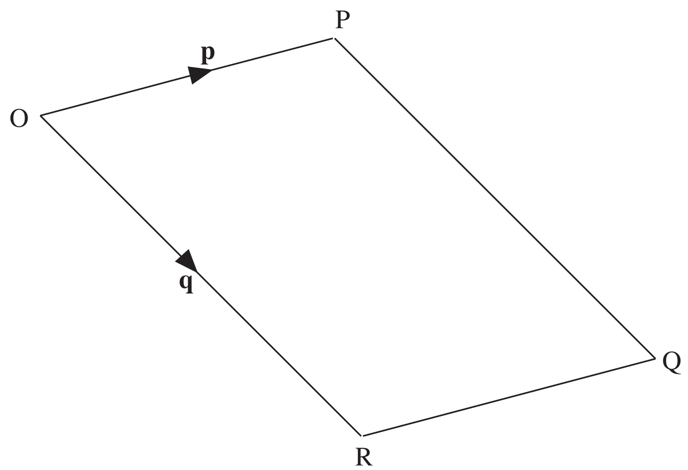
(iv) $\overrightarrow{AC}$ (v) $\overrightarrow{OB}$ (vi) $\overrightarrow{CA}$

(b) If X is the midpoint of AB and Y is the midpoint of BC, find the following in terms of **a** and **c**.

(i) $\overrightarrow{AX}$ (ii) $\overrightarrow{OX}$ (iii) $\overrightarrow{CY}$

(iv) $\overrightarrow{OY}$ (v) $\overrightarrow{XY}$

4. The diagram shows the parallelogram OPQR; the vectors $\mathbf{p} = \overrightarrow{OP}$ and $\mathbf{q} = \overrightarrow{OR}$.

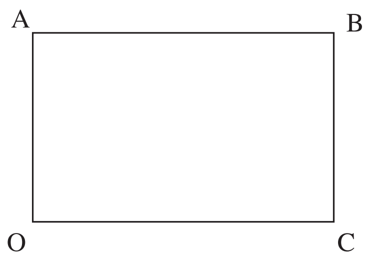


(a) If M is the midpoint of PR, find $\overrightarrow{OM}$.

(b) If N is the midpoint of OQ, find $\overrightarrow{ON}$

(c) Comment on your answers to (a) and (b).

5. The diagram shows the rectangle OABC. The vectors **i** and **j** are such that $\overrightarrow{OA} = 6\mathbf{j}$ and $\overrightarrow{OC} = 8\mathbf{i}$.

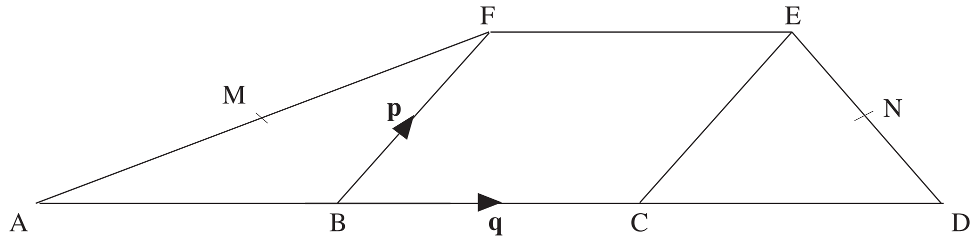


(a) If the point D lies on AB such that $AD = 3DB$, find $\overrightarrow{AD}$ and $\overrightarrow{OD}$ in terms of **i** and **j**.

(b) If E lies on BC such that $2BE = EC$, find $\overrightarrow{CE}$ and $\overrightarrow{OE}$.

(c) The point M is the midpoint of DE. Find $\overrightarrow{OM}$.

6. In the diagram, BCEF is a parallelogram and $AB = BC = CD$.
The vector $\mathbf{p} = \overrightarrow{BF}$ and $\mathbf{q} = \overrightarrow{BC}$. The point M is the midpoint of AF and N is the midpoint of DE.



(a) Express $\overrightarrow{AM}$ and $\overrightarrow{AN}$ in terms of **p** and **q**.

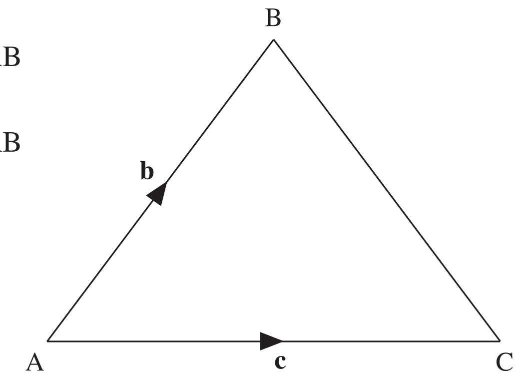
(b) Find $\overrightarrow{MN}$ in terms of **p** and **q** and explain why MN is parallel to AD.

7. In the triangle ABC, $\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{AC} = \mathbf{c}$.

Use vectors to show that:

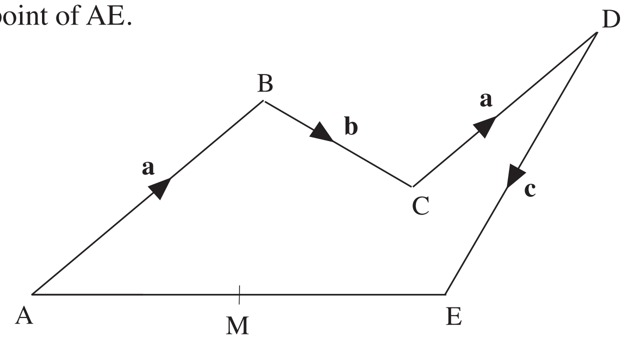
(a) a line joining the midpoint of AB and BC is parallel to AC,

(b) a line joining the midpoint of AB and AC is parallel to BC.



8. The shape in the diagram shows the vectors **a**, **b** and **c**.

M is the midpoint of AE.



Find each of the following in terms of **a**, **b** and **c**.

(a) $\overrightarrow{AE}$ (b) $\overrightarrow{AM}$ (c) $\overrightarrow{BA}$

(d) $\overrightarrow{MD}$ (e) $\overrightarrow{CM}$

9. ABCD is a parallelogram in which $\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{AD} = \mathbf{d}$.

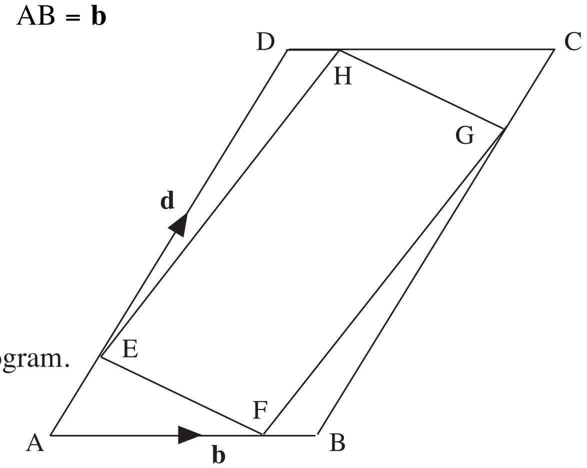
AF $= 4$BF

BG $= 4$CG

CH $= 4$DH

DE $= 4$AE

Show that EFGH is also a parallelogram.

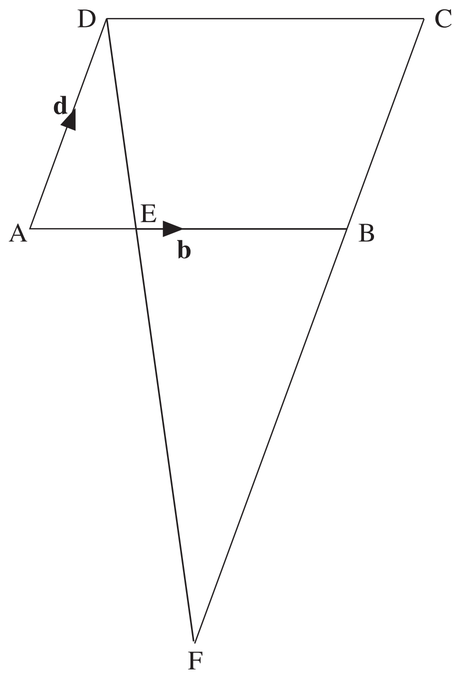


10. In the parallelogram ABCD, $\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{AD} = \mathbf{d}$. Also BE $= 2$AE.

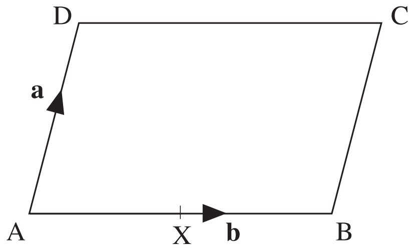
(a) Find $\overrightarrow{DE}$ and explain why $\overrightarrow{AF} = \mathbf{d} + \alpha\left(\frac{1}{3}\mathbf{b} - \mathbf{d}\right)$ for some values of α.

(b) Find $\overrightarrow{AC}$ and explain why $\overrightarrow{AF} = \mathbf{b} + \mathbf{d} - \beta\mathbf{d}$ for some value of β.

(c) Hence find the values of α and β.



11. In the parallelogram ABCD, $\overrightarrow{AD} = \mathbf{d}$ and $\overrightarrow{AB} = \mathbf{b}$. The point X is the midpoint of AB. The lines AC and DX intersect at Q.

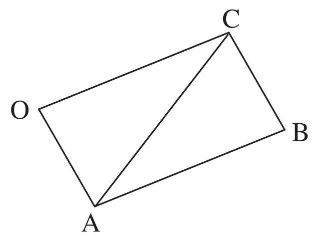


Find $\overrightarrow{AQ}$ in terms of **a** and **b**.

12. OABC is a parallelogram.

$\overrightarrow{OA} = 3\mathbf{p} - 2\mathbf{q}$

$\overrightarrow{OC} = 5\mathbf{p} + 6\mathbf{q}$



(a) Find $\overrightarrow{AC}$. Express your answer as simply as possible in terms of **p** and **q**.

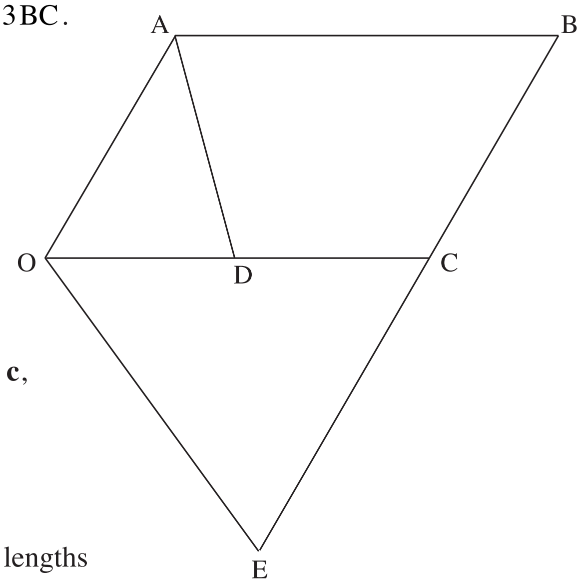
(b) D is the point where $\overrightarrow{BD} = -2\mathbf{p} + 6\mathbf{q}$.

Using vector methods, show that D lies on the line AC produced.

13. OABC is a parallelogram. $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OC} = \mathbf{c}$.

BCE is a straight line, $\overrightarrow{BE} = 3\overrightarrow{BC}$.

D is the midpoint of OC.



(a) Write in terms of **a** and **c**,

(i) $\overrightarrow{AD}$,

(ii) $\overrightarrow{OE}$.

(b) Deduce the ratio of the lengths of AD and OE.

14. *KLMN* is a parallelogram with position vectors

$$\vec{OK} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \vec{OL} = \begin{pmatrix} 6 \\ -3 \end{pmatrix} \text{ and } \vec{ON} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

(a) Use a vector method to determine the position vector $\vec{OM}$.

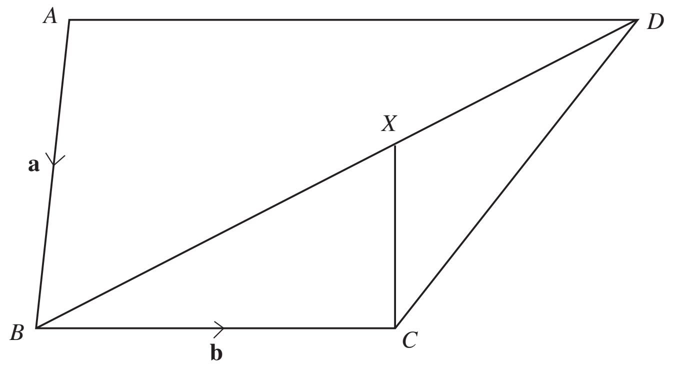
(b) The point *H* lies on *KM* such that $KH = HM$.

Find $\vec{KH}$ and $\vec{LH}$.

(c) Use a vector method to show that *H* is also the midpoint of *LN*.

(CXC)

15.



ABCD is a quadrilateral, **not drawn to scale**, with $\vec{AB} = \mathbf{a}$, $\vec{BC} = \mathbf{b}$, and $AD = 2\,BC$.

The point *X* divides *BD* in the ratio 3 : 2.

(a) Express $\vec{BD}$ and $\vec{BX}$ in terms of **a** and **b**.

(b) Show that $\vec{XC} = \frac{1}{5}(3\mathbf{a} - \mathbf{b})$.

(CXC)